The Logic of Logical Necessity

Andrew Bacon
Kit Fine*

May 23, 2022

Abstract

Prior to Kripke's seminal work on the semantics of modal logic, McKinsey offered an alternative interpretation of the necessity operator, inspired by the Bolzano-Tarski notion of logical truth. According to this interpretation, 'it is necessary that A' is true just in case every sentence with the same logical form as A is true. In our paper, we investigate this interpretation of the modal operator, resolving some technical questions, and relating it to the logical interpretation of modality and some views in modal metaphysics. In particular, we present an hitherto unpublished solution to problems 41 and 42 from Friedman's 102 problems, which uses a different method of proof from the solution presented in the paper of Tadeusz Prucnal.

A common conception of a logical truth, often credited to Bolzano, is that of a sentence true in virtue of its logical form alone. In a given interpreted language one might make this precise by stipulating a sentence to be logically true if and only if the result of uniformly substituting any of the non-logical constants with expressions of the same grammatical category is true, and dually, logically consistent if and only if some substitution instance is true. For instance, 'if John is tall then John is tall' is a logical truth, since the result of substituting any name and predicate for 'John' and 'is tall' respectively results in a truth, whereas 'John is tall' is not a logical truth because it is either already false or the result of substituting the predicate 'not tall' for 'tall' in it is false.

This analysis makes salient a formal analogy between the notion of a substitution in the Bolzanoean definition of logical truth and logical consistency, and the notion of a possible world in the analysis of necessity and possibility in a Kripke model. Indeed, prior to Kripke's work on the semantics of modal logic, McKinsey proposed a substitutional interpretation — or more accurately, a constraint on such an interpretation — of the modal operator exactly along Bolzanoean lines.[1] Given a language $\mathcal{L}$ containing the usual truth functional connectives and a unary connective $\Box$, McKinsey laid out some conditions that the set of true sentences of this language should satisfy, on the intended interpretation of $\Box$. Apart

*We cross generations in paying tribute to Saul's work on modal logic, one of us having learned of his work at its outset and the other having learned of it after several decades of development. He has done more than anyone to put the semantical study of modal logic on a firm technical footing, and our debt to his pioneering work should be evident throughout the paper.

Thanks are due to Lloyd Humberstone and Peter Fritz for correspondence with AB on this material that greatly benefited the paper. Theorems 13, 24 and 34 are due to AB, and theorems 9, 26, 42 and section 6 are due to KF. The results in section 4.2 were joint. Other minor propositions and observations are due to AB unless otherwise indicated.

[1]See McKinsey (1945).

from the usual clauses for the truth functional compounds, McKinsey requires that for every sentence A:

The Substitutional Constraint $\Box A$ is true if and only if iA is true for every substitution i of the language.

McKinsey leaves it open what sort of language $\mathcal{L}$ is, and the exact nature of the class of substitutions, stipulating only that the sentences of the language be closed under the truth-functional and modal connectives, and that the substitutions be closed under composition and contain a trivial substitution. We will consider more concrete versions shortly.

Under an alternative approach, the logical truths are not as Bolzano thought of them, but are nonetheless given by some theory Δ. Since Δ is being informally understood as the set of logical *truths* we should require that Δ be a subset of the truths, and one could then replace McKinsey's constraint with:

The Metalogical Constraint $\Box A$ is true if and only if $A \in \Delta$.

constraining the interpretation of the modal operator $\Box$ to include in its extension propositions expressed by sentences in Δ and to exclude propositions expressed by sentences not in Δ. Provided the theory Δ is closed under the rule of substitution, one direction of McKinsey's constraint is ensured, although not necessarily the other. In the case where $\mathcal{L}$ is the language of propositional modal logic, and Δ a logic (which we may assume to be at least closed under modus ponens and the rule of necessitation), we obtain an interpretation of propositional modal logic hit upon independently by Meyer and Fine in the 70s.[2]

Whether or not these constraints are plausible will depend both on the interpretation of the modal operator and on the range of interpretations of the non-logical constants of the language in which the constraints are formulated.

Under what interpretations of the modal operator could these schemas be plausible? One possibility, following Quine, is to treat the modal operator as 'crypto-quotational', so that a sentence embedded under a modal operator should be understood as residing inside invisible quotation marks, and the modality ascribes some metalinguistic property to the embedded sentence. But although we motivated the constraints by analogy with the metalinguistic notion of logical truth, one doesn't *have* to identify the modality under either constraint with a metalinguistic property. One could instead posit a genuine propositional operator that yields an interpretation of $\Box$ under which the substitutional or metalogical conditions are satisfied with respect to a suitable language $\mathcal{L}$. The schemas then impose a substantive constraint on a propositional operator by articulating a sense in which the postulated notion of logical necessity stands to the world as logical truth stands to language, without thereby identifying the two.

The constraints are not plausible on arbitrary interpretations of the non-logical constants either. For instance, if the language in question contained predicates 'bachelor' and 'married' with their customary meanings then either constraint will imply, given reasonable

[2]Specifically, the metalogical constraint is satisfied by any metavaluation (see Meyer (1971), and section 5.2 below): a valuation mapping sentences of propositional modal logic to 1 or 0, satisfying the conditions that $v(A \wedge B) = min(v(A), v(B))$, $v(\neg A) = 1 - v(A)$, and $v(\Box A) = 1$ iff $A \in \Delta$. If, additionally, $v(A) = 1$ whenever $A \in \Delta$ for any metavaluation v, Δ is called *coherent*. Fine applies the metalogical interpretation of $\Box$ to obtain simple proofs of the disjunction problem in modal and intuitionistic logics. For instance, any coherent modal logic, Δ, will have the disjunction property, for if $\Box A_1 \vee ... \vee \Box A_n \in \Delta$, then $v(\Box A_1 \vee ... \vee \Box A_n) = 1$ is true in any metavaluation v based on Δ, and so $A_i \in \Delta$ for some i. Indeed a similar argument may be used to establish that many logics possess an apparent strengthening of the disjunction property, which we call the extended disjunction property; the details may be found in section 5.2.

side conditions, that it is possible that there are married bachelors.[3] While this result might be acceptable when it is interpreted as a metalinguistic logical consistency claim in disguise, it is arguably not so on the worldly interpretation proposed above. For plausibly, to be a bachelor *just is* to be a man who is not married, so, by Leibniz's law, the alleged possibility would imply that it is possible that there are married men who are not married, which is certainly not true on any candidate interpretation of the modal operator.(Such an appeal to Leibniz's law would not be legitimate on a crypto-quotational reading of the modal operator as logical consistency, since Leibniz's law does not permit the substitution of identicals within quotation marks.)

One can avoid these untoward results by working in what Russell calls a *logically perfect* language: a language where the non-logical constants do not denote logically complex properties and no two non-logical constants codenote.[4] The former requirement rules out predicates like 'bachelor' that denote conjunctions of simpler properties, and the latter pairs of synonymous predicates like 'lawyer' and 'attorney'. The schemas then capture a sort of Humean vision in which the logically simple, or fundamental, properties and relations can, in the proposed sense of 'can', stand in any consistent logical relationship to one another.

Lastly, McKinsey left it open what sort of language could be plugged into his constraint. It could simply be the language of propositional modal logic — and this will be the option we will explore in this paper — although it could be a more expressive language, such as a first-order or higher-order language. If it is a higher-order language there is the possibility of a Tarskian analysis of logical truth. A sentence $A(c_1...c_n)$ in non-logical constants $c_1...c_n$ is a logical truth, according to Tarski, when it is true under every possible interpretation of $c_1...c_n$. We can achieve generality over the interpretations $c_1...c_n$ in a higher-order language by simply quantifying into the positions they occupy, leaving the logical truth of $A(c_1...c_n)$ amounting to the truth of $\forall x_1...x_n.A(x_1...x_n)$, where each x_i is a variable of the same type as c_i. Thus, for instance, the logical truth of 'If John is tall then John is tall' amounts to the mere truth of 'for any individual x and property Y if x is Y then x is Y'. In this case, we can fully internalize what in the first two constraints was stated in metalinguistic terms, since $\forall x_1...x_n.A(x_1...x_n)$ is a sentence of the object language. So another constraint, in the same spirit as the substitutional and metalogical constraints, replaces the Bolzanoean conception of logical truth with the Tarskian, yielding:

The Tarskian Constraint $\Box A(c_1...c_n) \leftrightarrow \forall x_1...x_n.A(x_1...x_n)$.

And this can likewise be thought of as articulating a Humean metaphysics of freely recombinable fundamental entities.[5]

The paper is organized as follows. In section 1, McKinsey's substitutional constraint on the interpretation of necessity is made precise within the context of propositional modal logic, and associated notions of valuation and validity are defined. In section 2 we raise the question of when an assignment of truth values to the sentence letters can be extended

[3]We can assume Δ is a *logical* theory, so it will not prove specific relationships concerning the non-logical constants, and we can also assume that the substitutions are rich enough we may substitute for 'married' and 'bachelor' predicates, like 'married' and 'not a bachelor', that are coinstantiated.

[4]Russell writes: 'In a logically perfect language the words in a proposition would correspond one by one with the components of the corresponding fact, with the exception of such words as "or", "not", "if", "then", which have a different function. In a logically perfect language, there will be one word and no more for every simple object, and everything that is not simple will be expressed by a combination of words, by a combination derived, of course, from the words for the simple things that enter in, one word for each simple component.' Russell (1940, p.25).

[5]See Bacon (2020).

uniquely to a valuation of the modal language satisfying the substitutional constraint. We show that unique extensions exist for any class of non-modal substitutions, and for the full substitution class we present a previously unpublished proof of the existence of an extension, leaving the uniqueness as a conjecture. Sections 3 and 4 concern the logic of logical necessity. We consider the validity of various principles of modal logic with respect to various substitution classes, such as the McKinsey and Grzegorczyk axioms, and settle some questions raised in Humberstone (2016). Section 5 treats the metalogical and Tarskian constraints, and the paper concludes with some remarks on the substitutional approach to modal predicate logic.

1 Substitutional interpretations of $\Box$

1.1 Preliminaries

McKinsey does not specify the language he is working in or provide a concrete account of what he means by a substitution. Rather, he proceeds by imposing some abstract constraints on the language and the substitutions. He assumes that the language is closed under the formation of truth-functional compounds and necessity sentences. For simplicity, we will assume that our own language $\mathcal{L}$ is at least closed under conjunction and negation and the formation of necessity sentences:

L1. If $A, B \in \mathcal{L}$ then $(A \wedge B) \in \mathcal{L}$

L2. If $A \in \mathcal{L}$ then $\neg A \in \mathcal{L}$

L3. If $A \in \mathcal{L}$ then $\Box A \in \mathcal{L}$

We will regiment McKinsey's account of substitution in terms of a set, S, of abstract substitutions and an action $\mu : S \times \mathcal{L} \to \mathcal{L}$ where $\mu(i, A)$ — written as iA — informally represents the result of applying the substitution $i \in S$ to the sentence A of $\mathcal{L}$ to produce another sentence of $\mathcal{L}$. We shall say that a pair (S, μ) is a *substitution class* if and only if the conditions below of Commutativity, Identity and Composition are satisfied. We will often suppress mention of the action μ, and refer to a substitution class solely by its associated set of substitutions provided no ambiguity arises.

Commutativity The action of each substitution $i \in S$ on $\mathcal{L}$ commutes with the logical connectives, $\wedge, \neg, \Box$:

1. $i(A \wedge B) = (iA \wedge iB)$
2. $i\neg A = \neg iA$
3. $i\Box A = \Box iA$

Identity There is a substitution $i \in S$ such that $iA = A$ for every sentence A of $\mathcal{L}$.

Composition For any two substitutions, $i, j \in S$ there is as substitution $k \in S$ such that $kA = i(jA)$ for every sentence A of $\mathcal{L}$.

As previously mentioned, these constraints may be satisfied by some very expressive languages, including first and higher-order languages. However we will mostly restrict our attention to propositional languages. Going forward, we will use $\mathcal{L}$ to refer to the propositional modal language with letters $p_0, p_1, p_2, ...$ and connectives $\wedge, \neg, \Box$. For technical reasons, it will sometimes be convenient to suppose that the language contains primitive 0-ary constants $\top$ and $\bot$, in which case the Commutativity condition must be extended to include the equations $i\top = \top$ and $i\bot = \bot$ for any $i \in S$. (Note that these equations might not hold if $\bot$, for instance, were treated as a defined connective, such as $(p_0 \wedge \neg p_0)$.)

Occasionally we will need to talk about other languages with further or fewer connectives. For any subset $\{C_1...C_n\} \subseteq \{\wedge, \vee, \neg, \rightarrow, \top, \bot, \Diamond, \Box\}$ we will write $\mathcal{L}(C_1...C_n)$ to represent the sublanguage in the connectives $C_1...C_n$: the smallest set containing the sentence letters, and containing $C_m A_1...A_k$ whenever it contains $A_1...A_k$, k the arity of C_m and $1 \leq m \leq n$. $\mathcal{L}()$ thus refers to the set of sentence letters. A substitution class for a language with some of these logical connectives is defined as above except that we require the substitutions commute with any additional logical connectives, as with $i(A \vee B) = (iA \vee iB)$ or $i\Diamond A = \Diamond iA$. We say that a language is *complete* iff it is truth-functionally complete and contains either $\Box$ or $\Diamond$.

Example 1 (The full monoid of substitutions of propositional modal logic)**.** *Let S be the set of concrete substitutions of $\mathcal{L}$, i.e. functions $i : \mathcal{L}() \rightarrow \mathcal{L}$ from sentence letters to arbitrary sentences of $\mathcal{L}$. $\mu(i, A)$ may be defined as the result of uniformly substituting $i(p_k)$ for p_k in A, for each $k \in \mathbb{N}$.*

Commutativity and L1-L3 are clearly satisfied. Identity is witnessed by the element that maps each letter to itself, and composition by the element of S that maps each letter p_k to $\mu(i, j(p_k))$.

Note, however, that a set satisfying McKinsey's requirements may not be isomorphic to the full monoid of substitutions of a language. There might be distinct substitutions $i, j \in S$ such that $iA = jA$ for every sentence $A \in \mathcal{L}$ (there could, for instance, be two identity elements satisfying Identity). Say that $i \sim j$ iff $iA = jA$ for every $A \in \mathcal{L}$. If one quotients a set S satisfying Commutativity, Identity and Composition by this equivalence relation one gets another set $S/\sim$ which satisfies Commutativity, Identity and Composition under the action defined by setting $\mu_\sim([i]_\sim, A) = \mu(i, A)$. Indeed, under this quotienting operation S forms a monoid: the unit ι may be defined as the equivalence class $[i]_\sim$ of any element of S satisfying Identity, and $[i] \circ [j]$ may defined to be the equivalence class of any k satisfying Composition. For most purposes we can simply treat S as a set of concrete substitutions of the language containing the identity substitution and closed under composition of substitutions. However, even if we impose these additional conditions, one still does not get that any *function* from the non-logical constants (the sentence letters in this case) to arbitrary expressions (sentences in this case) extends to a substitution in S. This property is distinctive to the full substitution class alone.

Example 2 (Substitutions within a sublanguage)**.** *Write $S(C_1...C_n)$ for the substitution class defined by the set of functions*

- $i : \mathcal{L}() \rightarrow \mathcal{L}(C_1...C_n)$

When $\mathcal{L}'$ is a language containing $C_1...C_n$ (so that $\mathcal{L}(C_1...C_n) \subseteq \mathcal{L}'$), the action of $i \in S(C_1...C_n)$ may be defined in the usual way (as in Example 1), thereby satisfying Commutativity.

Since $\mathcal{L}(C_1...C_n)$ contains each sentence letter, $S(C_1...C_n)$ contains the identity substitution and satisfies Identity. Since $\mathcal{L}(C_1...C_n)$ is itself a language, it is closed under substitutions of letters by sentences in $\mathcal{L}(C_1...C_n)$. So Composition is also satisfied.

We will also investigate a class of substitutions introduced in Humberstone (2016), in relation to McKinsey's theory of necessity:

Example 3 (Humberstone substitutions). *A Humberstone substitution is a function $i : \mathcal{L}() \to \mathcal{L}(\top\bot)$ such that*

- *$i(p_k)$ is either p_k, $\top$ or $\bot$.*

These substitutions act on the language with primitive $\top$ and $\bot$ connectives (i.e. on $\mathcal{L}(\neg\wedge\Box\top\bot)$). $\mu(i, A)$ is defined in the usual way, so that McKinsey's conditions are satisfied. We call the set of Humberstone substitutions H.

A positive formula is one whose letters all occur positively, where the letters occurring positively and negatively in a formula are defined by a simultaneous recursion as follows:

- $P(p_k) = \{p_k\}$, $N(p_k) = \emptyset$,
- $P(A \wedge B) = P(A) \cup P(B)$, $N(A \wedge B) = N(A) \cup N(B)$,
- $P(\neg A) = N(A)$, $N(\neg A) = P(A)$
- $P(\Box A) = P(A)$, $N(\Box A) = N(A)$.

Example 4 (Positive substitutions). *A positive substitution is a substitution that maps letters to positive formulas, and acts on formulas in the usual way. We will call the class of positive substitutions P.*

Lastly, we consider a special sort of substitution that will play a role later in our discussion of Carnap's theory of logical necessity.

Example 5 (Carnapian substitutions). *A Carnapian substitution is a function i on sentence letters such that*

- *$i(p_k)$ is the result of prefixing some number (possibly zero) of $\neg$ sign to p_k.*

and $\mu(i, A)$ is defined as before. We call the set of Carnapian substitutions K.

1.2 S-valuations, Pre-validity and Validity

We introduce the notion of an S-valuation: a valuation, taking sentences of the language $\mathcal{L}$ to truth values, that satisfies McKinsey's constraints.

Definition 1. *Suppose that $\mathcal{L}$, S and μ satisfy L1-L3, Commutativity, Identity and Composition. An S-valuation is a function $v : \mathcal{L} \to \{0, 1\}$ such that*

- *$v(A \wedge B) = min(v(A), v(B))$*
- *$v(\neg A) = 1 - v(A)$*
- *$v(\Box A) = 1$ if and only if, for every $i \in S$, $v(iA) = 1$*

By a truth-value assignment *we mean a function defined on the sentence letters, $v^- : \mathcal{L}() \to \{0, 1\}$. A valuation v extends a truth-value assignment, v^-, iff $v \restriction_{\mathcal{L}()} = v^-$.*

Formally speaking we identify truth and falsity with the numbers 1 and 0, but we shall sometimes talk of a sentence being true or false in a valuation rather than receiving the value 1 or 0.

In cases where some of $\top, \bot, \vee$ or $\Diamond$ are also taken as primitive, the notion of valuation should respect the natural clauses for those connectives:

- $v(A \vee B) = max(v(A), v(B))$
- $v(\top) = 1$
- $v(\bot) = 0$
- $v(\Diamond A) = 1$ if and only if, for some $i \in S$, $v(iA) = 1$.

We introduce two senses in which a formula can be valid relative to a substitution class. The first is not so straightforwardly related to the Bolzano-Tarski conception of validity, since the set of valid sentences in this sense need not be closed under the rule of substitution:

Definition 2 (Pre-validity)**.** *A sentence A is pre-valid relative to the substitution class S if and only if $v(A) = 1$ for every S-valuation v.*

To see that pre-validity is not closed under the rule of substitution, consider any substitution class S which contains a substitution i such that $i(p_0) = \top$ (as with $S(C_1...C_n)$ for any set of connectives $C_1...C_n$ containing $\top$). For any S-valuation v, $v(\Diamond p_0) = 1$ because $v(ip_0) = v(\top) = 1$. So $\Diamond p_0$ is pre-valid relative to the class S. However the substitution instance $\Diamond\bot$ is not pre-valid; indeed it is false in every S-valuation.

Validity proper thus cannot be understood as the result of closing the pre-validities under the rule of substitution. Rather we consider a sentence to be valid relative to a substitution class S only if all of its substitution instances are pre-valid.

Definition 3 (Validity)**.** *A sentence A is valid relative to the substitution class S if and only if iA is pre-valid relative to the substitution class S, for every substitution i in the full substitution class $S(\neg \wedge \Box)$.*

If S is a substitution class, then we will write $L(S)$ for the set of sentences valid with respect to S, and we call it the logic of *S.*

Note that the 'external' notion of validity defined above, for which we required the theorems of the logic to be valid, can be distinguished from the 'internal' notion of validity by which $\Box$ is interpreted. The former requires truth under all substitutions, whereas the latter requires only truth under all substitutions in S. Consequently, they can diverge when $\Box$ is interpreted by a notion of validity more restricted than our external notion of logical validity. According to the interpretation of $\Box$ as *logical* validity, however, the two notions will coincide, and for this reason the case where S is the full substitution class is of special interest.[6]

Let us remark briefly on the choice of primitives. It is easily verified that standard definitions of arbitrary truth-functional compounds in terms of $\neg$ and $\wedge$ receive the same

[6] It is also possible to adopt an external notion of validity intermediate between prevalidity and validity, whose validities are those sentences with prevalid S-substitution instances. This secures the coincidence between the external and internal notions in another way, although this is not an avenue we have pursued. The rationale behind requiring logic to be closed under arbitrary substitutions, at any rate, is that logic shouldn't say anything substantive about particular individuals, properties or relations (apart from the logical ones), so that any sentence that holds, as a matter of logic, for a particular non-logical constant should hold, as a matter of logic, for an arbitrary expression of the same type.

truth value in any valuation. Similarly $\Diamond$ may be defined, as usual, in terms of $\neg$ and $\Box$. In most contexts it is more economical to work in the language $\mathcal{L}$, whose primitives consist of only $\wedge, \neg$ and $\Box$. But in some cases, the presence of the other primitives make the definitions simpler: for instance if we identified $\top$ and $\bot$ with particular definitions, such as $\neg(p_0 \wedge \neg p_0)$ and $p_0 \wedge \neg p_0$, then our official definition of $S(\top\bot)$ would not satisfy the commutativity equations $i\bot = \bot$ and $i\top = \top$.

The differences between the logics of different substitution classes will be the subject of the next few sections. However, we are already in a position to see how different substitution classes may behave differently at the level of pre-validity. We have, for instance, the following contrast between full and non-modal substitutions:

Proposition 1. *The sentence $\Diamond(\Diamond p \wedge \Diamond\neg p \wedge \Box(p \to \Box p))$ (for p a sentence letter) is pre-valid over $S(\neg \wedge \Box)$, while its negation is pre-valid over $S(\neg\wedge)$.*

Proof. Let v be any $S(\neg \wedge \Box)$ valuation. First note that $v(\Box\top) = 1$ since $v(i\top) = v(\top) = 1$ for every $i \in S(\neg \wedge \Box)$. Similarly for any sentence letter q, $v(\neg\Box q) = 1$: $v(\Box q) = 0$ since $v(r \wedge \neg r) = 0$ and for some $i \in S(\neg \wedge \Box)$, $iq = r \wedge \neg r$.

We shall show that upon substituting $\Box q$ for p in $(\Diamond p \wedge \Diamond\neg p \wedge \Box(p \to \Box p))$ we get something true in v, and so $\Box q$ witnesses the truth of $\Diamond(\Diamond p \wedge \Diamond\neg p \wedge \Box(p \to \Box p))$. (i) $v(\Diamond\Box q) = 1$ since $\Box q$ has a true-in-v substitution, namely $v(\Box\top) = 1$. (ii) $v(\Diamond\neg\Box q) = 1$ since $\neg\Box q$ is already true in v. (iii) $v(\Box(\Box q \to \Box\Box q)) = 1$. Take any $i \in S(\neg \wedge \Box)$. If $v(\Box iq) = 1$ then $v(jiq) = 1$ for every j. This means $v(kliq) = 1$ for any k and l, and so $v(\Box\Box iq) = 1$. Thus $v(\Box iq \to \Box\Box iq) = 1$ for any $i \in S(\neg \wedge \Box)$, and so $v(\Box(\Box q \to \Box\Box q) = 1$.

Now let v be any $S(\neg\wedge)$-valuation, take any $i \in S(\neg\wedge)$, and let $A := ip$. By definition A is non-modal. Suppose for reductio that $v(\Diamond A \wedge \Diamond\neg A \wedge \Box(A \to \Box A)) = 1$. Since $v(\Diamond A) = 1$ and A is non-modal, there is some valuation of propositional logic, u, on the non-modal language that makes A true. Now construct a substitution j that maps each letter p in A to a literal that is true in v, if $u(p) = 1$, and a literal that is false in v if $u(p) = 0$, and in such a way that we never pick two literals containing the same letter. Since for each letter p_m occuring in A we have $v(jp_m) = u(p_m)$, we have that $v(A(jp_1...jp_n)) = u(A(p_1...p_n)$ (writing A as a function of its sentence letters), which means $v(jA) = u(A) = 1$. Moreover, since $v(\Diamond\neg A) = 1$, A is not a tautology, and so jA is not either (from the way it was constructed). So $v(\Box jA) = 0$ (since if B is non-modal, $v(\Box B) = 1$ only if B is a tautology). So we have $v(jA) = 1$ and $v(\Box jA) = 0$ so $v(jA \to \Box jA) = 0$ and so $v(\Box(A \to \Box A)) = 0$. This contradicts the assumption that $v(\Diamond A \wedge \Diamond\neg A \wedge \Box(A \to \Box A)) = 1$.

Thus $v(\Diamond ip \wedge \Diamond\neg ip \wedge \Box(ip \to \Box ip)) = 0$ for every non-modal substitution i; and so $v(\Diamond(\Diamond p \wedge \Diamond\neg p \wedge \Box(p \to \Box p))) = 0$ for every $S(\neg\wedge)$-valuation v. $\Box$

This proposition technically doesn't rule out the pre-validities of $S(\neg\wedge)$ and $S(\neg \wedge \Box)$ coinciding by being the inconsistent logic. We will eventually rule out this possibility by showing the existence $S(\neg\wedge)$ and $S(\neg \wedge \Box)$-valuations. It's also worth noting, however, that despite the stark contrast between the pre-validities of these two classes, for all we know the logics of these two substitution classes are identical.

1.3 Relationship to Kripke models

We briefly remark on the relationship between the foregoing substitutional interpretation of propositional modal logic and the more familiar interpretation due to Kripke (1959). Although the clauses for substitutional necessity are on the surface quite different from the clauses for necessity in a Kripke model, any S-valuation may be reconceived as a Kripke model.

Definition 4. *For any given substitution class, S, the* Kripke frame associated with S, *(W_S, R_S), is defined by:*

- $W_S := S$
- $R_S := \{(i, j \circ i) \mid i, j \in S\}$, *that is, $R_S ik$ if and only if there is some $j \in S$ such that $k = j \circ i$.*

Given an S-valuation, v, the Kripke model associated with v *is obtained by defining a valuation $V : W_S \times \mathcal{L}() \to \{0, 1\}$ on (W_S, R_S) as follows:*[7]

- $V(i, p_k) = v(ip_k)$.

V may then be extended to arbitrary formulas in the usual manner.

In what follows we reserve the lower case letters v and u for S-valuations and use the uppercase letters V and U for valuations in a Kripke model.

Proposition 2. *For any $i \in S$ and formula A, $V(i, A) = v(iA)$.*

Proof. By induction. The clauses for the sentence letters and truth functional connectives are trivial. For the modal clause, we reason as follows.

$V(i, \Box A) = 1$ if and only if, for every $j \in S$, $V(j \circ i, A) = 1$. By the inductive hypothesis, this holds iff $v(jiA) = 1$ for every $j \in S$, which holds iff $v(i\Box A) = 1$, as required. $\Box$

We will also frequently appeal to the notion of p-morphism of frames:

Definition 5 (p-morphism)**.** *A p-morphism from a Kripke frame (W, R) to another Kripke frame (W', R') is a function $f : W \to W'$ such that:*

(i) For any $x, y \in W$, if Rxy then $R'f(x)f(y)$.

(ii) For any $x \in W$ and $y' \in W'$, if $R'f(x)y'$ then there exists a $y \in W$ such that Rxy and $f(y) = y'$.

If f is a p-morphism between the frames of the models (W, R, V) and (W', R', V') and f preserves the truth values of some sentences, $B_1 ... B_n$, — i.e. $V(x, B_k) = V'(f(x), B_k)$ for $k = 1...n$ and any $x \in W$ — then it may be shown that for any sentence A constructed from $B_1 ... B_n$ using only the truth-functional and modal connectives, $V(x, A) = V'(f(x), A)$. If f preserves the truth values of all the letters (and so preserves the truth values of all sentences), then f is sometimes called a p-morphism between the models (W, R, V) and (W', R', V'). Given the above correspondence between valuations and Kripke models, we can talk about p-morphisms between substitution classes and Kripke frames.

Definition 6 (p-morphism between substitution classes and Kripke frames)**.** *A p-morphism from a substitution class S to a Kripke frame (W', R') is a p-morphism from the frame (W_S, R_S) to (W', R').*

[7] See Drake (1962).

2 The Existence and Uniqueness of S-valuations

If there were no valuations for a given substitution class, S, the concepts of pre-validity and validity would be trivial and uninteresting. However, given a substitution class S, the existence of an S-valuation is not always obvious. Consider, for instance, a valuation for the full substitution class mapping letters to arbitrary sentences of $\mathcal{L}$. While the truth values of conjunctions and negations are determined by the truth values of sentences of lower complexity (the conjuncts or the negatum), the truth value of modal formulas is not. The truth value of $\Box A$ is determined by the truth values of iA for all possible substitutions of letters within A, including substitutions of higher complexity: i might map some of the letters appearing in A to A itself for example, yielding a potentially vicious circularity.[8]

In many cases the circularity is not vicious. Let's say that a substitution i is non-modal iff ip_k is a formula of the propositional calculus for each k. The following proposition shows that whenever S is a class of non-modal substitutions, S-valuations exist and can be constructed in a familiar inductive manner. Indeed, any assignment to the sentence letters extends to a unique S-valuation on $\mathcal{L}$.

Proposition 3. *Suppose that S is a class of non-modal substitutions and that $v^- : \mathcal{L}() \to \{0,1\}$ is a truth-value assignment on the sentence letters of $\mathcal{L}$. Then there exists a unique S-valuation extending v^-.*

Proof. v may be defined inductively on the modal degree of the the formula. In particular, suppose that $v(A)$ has been defined for every formula of modal degree n, and we are attempting to evaluate $v(\Box A)$ and $v(\Diamond A)$ of modal degree $n+1$. Since each $i \in S$ is non-modal, iA will have modal degree n, and so $v(\Box A)$ and $v(\Diamond A)$ may and must be defined as $\min_{i\in S} v(iA)$ and $\max_{i\in S} v(iA)$, respectively. $\Box$

When S contains substitutions with modal formulas in their range, the constraints of Definition 1 can no longer be met through a straightforward inductive construction. Nonetheless, we might conjecture that an analogue of Proposition 3 holds for the full substitution class:

Conjecture 4 (The Uniqueness Conjecture)**.** *Suppose that S is the class of all substitutions on $\mathcal{L}$, and suppose that $v^- : \mathcal{L}() \to \{0,1\}$ is a truth-value assignment on the sentence letters. Then there exists a unique S-valuation v extending v^-.*

Indeed this conjecture is a slight variant of a conjecture made by Harvey Friedman in Friedman (1975).[9] The existence portion of the conjecture was settled independently by Kit Fine and Tadeusz Prucnal.[10] Prucnal's method is rather indirect, and goes via a solution to a related problem concerning intuitionistic logic. In section 2.1 we present Fine's direct proof of the existence of a valuation of the full substitution class. The uniqueness part of the conjecture is still open.

For some classes of modal substitutions we can prove the existence of an S-valuation through a slightly different inductive construction. To illustrate, consider the 'positive substitutions' from Example 4, that map formulas to positive formulas (in which letters only

[8]See the related remarks in Kripke (1976, p.332) concerning the substitutional interpretation of the quantifiers.

[9]Friedman's conjecture (see problem 42 of Friedman (1975)) was concerned with the existence and uniqueness of a valuation on the full substitution class making all the sentence letters true. Moreover, his version of the conjecture postulates the existence and *non*-uniqueness of such a valuation.

[10]See Prucnal (1979).

appear under an even number of negations). Let P be the class of all positive substitutions on $\mathcal{L}$.

Proposition 5. *There exist P-valuations.*

Proof. Let X_0 be the set of positive formulas, and X_{n+1} the truth functional combinations of formulas from $\{A, \Box A \mid A \in X_n\}$. We define a valuation inductively as follows. Each formula of X_0 is treated as true. For any positive substitution i and sentence $A \in X_n$, it is readily shown that iA is also in X_n. Thus assuming the valuation is defined on X_n we can extend the valuation (uniquely) to formulas of the form $\Box A$ for $A \in X_n$ and thus to arbitrary formulas in X_{n+1}.

$\Box$

2.1 The existence of valuations of the full substitution class

Next we settle the harder question of whether there are any valuations for the full substitution class. We will show that every truth-value assignment to the sentence letters, $v^- : \mathcal{L}() \to \{0,1\}$, can be extended to a valuation of the full substitution class.

In order to do this we will introduce a special class of finite, partially ordered frames that have the property that every world can easily be characterized, in relation to the endpoints of the order, by a modal formula. Henceforth we will write $P^0(X)$ for $P(X) \setminus \emptyset$ and we will write (being somewhat sloppy) $\supseteq$ for the superset relation restricted to $P^0(X)$ when X is clear from context.

Definition 7 (Medvedev frame)**.** *A Medvedev frame is a frame of the form $(P^0(X), \supseteq)$ for some finite non-empty set X.*

We write Med *for the logic of Medvedev frames.*

We begin with a brief informal overview of our proof strategy. The key idea is to take a countable set of disjoint models over Medvedev frames for which Med is complete, and glue them together to make a new model M by placing a new root world that lies underneath the original 'component' models and that sees itself and all the worlds of the component models. The truth values of the letters at the new root world are determined by v^-, and at the other worlds by their truth value in the relevant component model. We then argue that a sentence $\Box A$ is true at the root of this global model if and only if every substitution instance of A is true at the root. The valuation of the full substitution class is then obtained by identifying the truth value of a sentence with its truth value at the root world.

To make this precise, enumerate all of the consistent sentences of Med, $C_1, C_2, C_3, \ldots$. For each C_i, pick a model $M_i = (P^0(X_i), \supseteq, V_i)$ on which C_i is satisfiable at some world. We may assume without loss of generality that X_i and X_j are disjoint when $i \neq j$. Then our global model $M = (W, \supseteq, V)$ may be defined as follows:

- $X := \bigcup_{i \in \omega} X_i$
- $W := \{w \subseteq X \mid w = X \text{ or } w \in P^0(X_i) \text{ for some } i\}$
- $V(w, p_j) = V_i(w, p_j)$ if $w \in P^0(X_i)$ for some i
- $V(X, p_j) = v^-(p_j)$.

Our goal is to show that for any formula A:

Proposition 6. *$V(X, \Box A) = 1$ if and only $V(X, iA) = 1$ for every substitution i.*

The proof of the left-to-right direction is quite straightforward. The frame of M is not Medvedev because W is infinite, but it may nonetheless be shown to be a model of Med in the sense that $V(w, B) = 1$ for any formula $B \in \mathsf{Med}$ and world $w \in W$ (Lemma 7 below). The left-to-right direction follows, since if $\Box A$ is true at the root X of M, A must be true in each component model, and thus must be in Med by the completeness of Med with respect to the component models. But since Med is closed under the rule of substitution, every substitution instance of A is in Med, and so every substitution instance of A must be true at the root world since M is a model of Med.

The proof of the right-to-left direction is more involved. It suffices to show that if $\Box A$ is false at the root world X then some substitution instance of A is also false at X. If $\Box A$ is false at the root, then A must be false at some world in M, and so must not belong to Med since, as before, M is a model of Med. So A can be made false at a world in a model $N = (P^0(Y), \supseteq, U)$ over a Medvedev frame based on the set $Y = \{1, ..., n\}$; indeed it can be made false at the root Y of such a model.[11] This model refuting A encodes in a natural way the required substitution instance of A that is false at the root X of M. We now describe how to extract that substitution from the model N.

Without loss of generality we may assume that the terminal worlds of N, $\{1\}, ..., \{n\}$, make different propositional letters true and so there are propositional formulas $A_1, ...A_n$ such that A_m is true at $\{m\}$ and false at $\{m'\}$ for $m' \neq m$. For reasons that will come in to play later, we will also choose $A_1...A_n$ so that they are pairwise incompatible and have a tautologous disjunction.[12] Since a singleton $\{m\}$ sees only itself, $\Box A_m$ will be true at $\{m\}$. Now a given world Z (i.e. subset of $\{1, ..., n\}$), is uniquely determined by its singleton subsets, or equivalently the terminal worlds $\{m\}$ it sees. Thus Z makes $\Diamond\Box A_m$ when $m \in Z$ and $\Diamond\Box A_m$ false when $m \notin Z$, and is the unique world of our model to do so. Thus we may define a formula, D_Z that is true only at world Z in N:

$$D_Z := \bigwedge_{m \in Z} \Diamond\Box A_m \wedge \bigwedge_{m \notin Z} \neg\Diamond\Box A_m$$

Since N is finite, any set of worlds is characterized by a disjunction of these world formulas. In particular, for each propositional letter p we may define a formula made only out of formulas characterizing the terminal worlds (i.e. $A_1...A_n$) that has the same truth value as p at any world.

$$D_p := \bigvee_{U(Z,p)=1} D_Z$$

We can now define our desired substitution by mapping each letter p to D_p:

$$i(p) := D_p$$

Since p and ip have the same truth value at every world in N, the result of uniformly substituting p for ip in any formula must have the same truth value at any world in N. Since A is false at the root $Y = \{1, ..., n\}$ in N, iA must also be false at the root Y.

[11] If A is refuted at world Y of cardinality n at a non-root world of a larger Medvedev frame, the submodel generated by Y is easily seen to be isomorphic to a model on the Medvedev frame $\{1, ..., n\}$.

[12] This may always be done, for when we are constructing a model that refutes A over a Medvedev frame we are free to set the truth-values of letters not appearing in A how we please. So pick sentence letters $p_1...p_k$ not appearing in A such that $2^k \geq n$ and pick a surjection $\sigma : 2^k \to n$. A_m may then be defined as $\bigvee_{\sigma(u)=m}(\bigwedge_{u(r)=1} p_r \wedge \bigwedge_{u(r)=0} \neg p_r)$. A valuation in which $U(\{m\}, A_m) = 1$ can be constructed by picking some representative element u from $\sigma^{-1}(m)$ and letting $U(\{a_m\}, p_r) = u(r)$ for $r = 1...k$.

We now show that iA must also be false at the root of M. According to any world w in our global model M, some of $A_1...A_n$ are possibly necessary, and others are not: so there is a *unique* world in N that agrees with w about which $A_1...A_n$ are possibly necessary, namely the world $\{m_1, ..., m_k\}$ assuming $A_{m_1}, ..., A_{m_k}$ are possibly necessary at w in M. Note that the root of M, X, gets associated with $Y = \{1, ..., n\}$ since each A_k is possibly necessary at the root of M. Let us call this function associating worlds of M to worlds on N, $f : W \to P^0(Y)$:

$$f(w) = \{m \mid V(w, \Diamond\Box A_m) = 1\}$$

It may be shown (Lemma 8) that this function f is a p-morphism of frames (it is here we use the fact that $A_1...A_n$ are pairwise incompatible). Now if a p-morphism preserves the truth values of some modal formulas $C_1, ..., C_k$ then it will also preserve the truth value of any formula constructed from $C_1...C_k$ using the truth-functional and modal connectives (recall Definition 5). Since by definition f preserves the truth values of formulas of the form $\Diamond\Box A_k$, and the formulas D_p, and thus also iA, are constructed from these formulas using the modal and truth-functional connectives f preserves the truth values of the formulas D_p and thus also iA. Since f maps X to Y and iA is false at Y, iA is false at the root of M as required.

It remains to prove the lemmas appealed to above. The first is that M is a model of Med:

Lemma 7. *If $A \in \mathsf{Med}$ and $w \in W$, then $V(w, A) = 1$.*

This is a consequence of a more general result — Proposition 39 — that will we prove later in section 5.2, so we will defer its proof.[13]

Lemma 8. *f is a p-morphism from the frame of M to the frame of N*

Proof. We first we show that f preserves the accessibility relation. Suppose w sees v in M. Since the accessibility relation is transitive, anything possible at v is possible at w. So $f(v) \subseteq f(w)$ and hence $f(w)$ sees $f(v)$ in N.

To establish the reverse condition, suppose that $f(w)$ sees a world $Z \in P^0(Y)$, so that $Z \subseteq f(w)$. We must show that there exists a $v \subseteq w$ such that $f(v) = Z$.

By definition, for each $m \in f(w)$, $w \Vdash \Diamond\Box A_m$. Thus for each $m \in f(w)$ there is a terminal world $\{a_m\} \in W$ such that $\{a_m\} \subseteq w$ and $\{a_m\} \Vdash A_m$. Let $v = \{a_m \mid m \in Z\}$. Clearly $v \subseteq w$. Moreover, we may show that $f(v) = Z$. For, given any $m \in Z$, $\{a_m\} \Vdash \Box A_m$ and $\{a_m\} \subseteq v$, so $v \Vdash \Diamond\Box A_m$, and so $m \in f(v)$. Conversely, if $v \Vdash \Diamond\Box A_m$ then v sees a terminal world that makes A_m true. Since $A_1...A_n$ are pairwise incompatible the only singleton v sees where A_m is true is $\{a_m\}$, so $a_m \in v$ and so $m \in Z$. $\Box$

We may, finally, obtain a valuation v of the full substitution class by setting $v(A) = V(X, A)$.

Theorem 9. *Every truth-value assignment v^- may be extended to a valuation of the full substitution class.*

It should be observed that, in the constructed valuation, the truth of a possiblity sentence $\Diamond A$, may always be witnessed by a substitution of formulas of modal degree 2.

[13]In section 5.2 it is shown that Med is a *coherent logic* (see Definition 10), and Lemma 7 follows from the observation that the function v, defined by setting $v(A) := V(X, A)$, is a meta-valuation (see Definition 9).

2.2 The uniqueness conjecture

Proposition 3 established that for any non-modal substitution class S, any truth assigment to the sentence letters extends to a *unique* S-valuation. The analogue of Proposition 3 for the full substitution class is the principle we called Conjecture 4:

> Any truth-value assignment $v^{-} : \mathcal{L}() \to \{0, 1\}$ extends to a unique $S(\neg \wedge \Box)$-valuation.

Theorem 9 establishes the existence portion of the conjecture. The uniqueness conjecture is important to the study of logical necessity because it is implies that there are interpretations of $\Box$ in which $\Box$ means *valid*. Letting S be the full substitution class, Conjecture 4 is equivalent to:

> For any S-valuation v, $v(\Box A) = 1$ iff A is valid with respect to S.

Supposing the uniqueness conjecture to be true, $v(\Box A) = u(\Box A)$ for any pair of S-valuations, and so $v(\Box A) = 1$ iff $u(\Box A) = 1$ for all valuations u. Either side of the biconditional holds iff, for every valuation u and substitution i, $u(iA) = 1$. So under the supposition, $\Box A$ will coincide with validity with respect to the full substitution class. The converse holds too. For suppose the previous claim holds. Then for any pair of S-valuations u and v, $u(\Box A) = 1$ iff A is valid with respect to S iff $v(\Box A) = 1$; and so if v and u additionally agree on the sentence letters, then $u = v$.

Notice that in the valuation constructed in Theorem 9, $v(\Box A) = 1$ if and only if $A \in \mathsf{Med}$. Thus an extension of a truth-value assignment, v^{-}, to a valuation for the full substitution class may be defined inductively as follows.

- $v(p) = v^{-}(p)$
- $v(A \wedge B) = min(A, B)$
- $v(\neg A) = 1 - v(A)$
- $v(\Box A) = 1$ iff $A \in \mathsf{Med}$

For an arbitrary valuation v of $\mathcal{L}$, let $\Delta_v = \{A \mid v(\Box A) = 1\}$. Then another way to state Conjecture 4 is:

> For any valuation v of the full substitution class, $\Delta_v = \mathsf{Med}$.

While we are not able to settle the uniqueness conjecture, we will here establish the weaker claim:

> For any valuation v of the full substitution class, $\Delta_v \subseteq \mathsf{Med}$.

Moreover, in section 4.2 we'll show that some principles distinctive to Med must also belong to Δ_v. Thus Δ_v would appear to be tightly sandwiched between Med and a strong subsystem of Med, lending further credence to the uniqueness conjecture.

Let v be a valuation for the full substitution class. For each n, pick a propositional partition $A_1 ... A_n$ — i.e. n propositional formulas that are pairwise inconsistent and have a tautologous disjunction. For any substitution, i, let $f(i) = \{m \mid v(\Diamond \Box i A_m) = 1\}$. By a $\top\bot$ substitution we shall mean a substitution whose range on $\mathcal{L}()$ is $\{\top, \bot\}$.

Lemma 10. *For any substitution i and $Y \subseteq f(i)$, there exists a substitution j such that $f(j \circ i) = Y$.*

Proof. In this proof we will frequently appeal to the fact that if B is a formula, v a valuation, and j a substitution such that $v(\Box jB) = 1$, then there exists a $\top\bot$ substitution k such that $v(\Box kB) = 1$. For if $v(\Box jA) = 1$, for an arbitrary j then the result of pre-composing j with any $\top\bot$ substitution k', yields $\top\bot$ substitution, $k' \circ j$, with the required property.

Suppose that $Y \subseteq f(i)$. Let $p_1...p_r$ be the letters appearing in $A_1...A_n$, and henceforth let k be variable for a $\top\bot$ substitution defined only on $p_1...p_r$.

Since, by definition, $v(\Diamond\Box iA_m) = 1$ for each $m \in f(i)$, there must exist for each $m \in f(i)$ a substitution k such that $v(\Box kiA_m) = 1$. And without loss of generality we may assume that k is a $\top\bot$ substitution. For each $m \in f(i)$ we shall pick a representative $\top\bot$ substitution, k_m, such that $v(\Box kiA_m) = 1$.

Let S be the set of all $\top\bot$ substitutions defined on $p_1...p_r$, and let $S' = \{k_m \mid m \in Y\}$. Our goal will be to find a substitution j that 'filters' the substitutions in S leaving all and only substitutions in S' behind: i.e. for any $k \in S$, $k \circ j$ is equivalent to a substitution in S' and every substitution in S' is equivalent to something of the form $k \circ j$ for some $k \in S$. Here two substitutions, k and k' are said to be equivalent when $v(\Box(kp \leftrightarrow k'p)) = 1$ for any sentence letter p from $p_1...p_r$.

Because $A_1...A_n$ are pairwise inconsistent and have a tautologous disjunction, we also know that for each $\top\bot$ substitution k there is a unique sentence A_m such that $v(\Box kiA_m) = 1$. For a give k we will call this sentence A^k. Now pick any surjection $\sigma : S \to S'$ and define a substitution j as follows:

$$j(p) = \bigvee_{\sigma(k)(p)=\top} A^k$$

for each $p \in \{p_1...p_r\}$

We now show that for any $\top\bot$ substitution k, $k \circ j$ and $\sigma(k)$ are equivalent. Firstly, we show that for any $\top\bot$ substitution k on $p_1...p_n$, $\sigma(k)$ and $k \circ j$ are equivalent substitutions. We will show for each letter p, $v(\Box kjp) = 1$, if and only if $\sigma(k)(p) = \top$. Suppose that $\sigma(k)(p) = \top$. So A^k is a disjunct of $\bigvee_{\sigma(k)(p)=\top}(A^k) = j(p)$, and so $k(A^k)$ — which recall is necessary in v by definition — is a disjunct $kj(p)$. Conversely, suppose that $v(\Box kjp) = 1$. Since the A^k are pairwise incompatible, this can only happen if A^k is one of the disjuncts of $j(p)$, and thus can only happen if $\sigma(k)(p) = \top$.

This delivers us our desired result: for if $m \in Y$ then, because σ is surjective, there is some $\top\bot$ substitution such that $\sigma(k) = k_m$, and k_m and $k \circ j$ are equivalent. Since $v(\Box k_m iA_m) = 1$, by definition, and k_m is equivalent to $k \circ j$, $v(\Box kjiA_m) = 1$, and thus $v(\Diamond\Box jiA_m) = 1$. So $m \in f(j \circ i)$. Conversely if $m \in f(j \circ i)$, this means $v(\Diamond\Box jiA_m) = 1$ and so there is substitution k (which we may assume without loss of generality to be $\top\bot$) such that $v(\Box kjiA_m)$. $k \circ j$ is equivalent to $\sigma(k)$ which may be seen to equivalent to k_m. Since $v(\Box k_m iA_m) = 1$, $m \in Y$.

$\Box$

The significance of Lemma 10 is this. Let (W, R, V) be the Kripke model associated with v (so W is the class of substitutions, $R = \{(i, j \circ i) \mid i, j \in W\}$ and $V(i, p) = v(ip)$). Then:

Corollary 11. *$f : W \to P^0(X)$ is a p-morphism of frames from (W, R) to $(P^0(X), \supseteq)$ where $X = \{1, ..., n\}$.*

Proof. Since Rik iff there exists a j such that $k = j \circ i$, the back-and-forth condition amounts to the claim that if $f(i) \supseteq Y$ then there exists a j such that $f(j \circ i) = Y$. This is just Lemma 10.

It remains to show that $f(j \circ i) \subseteq f(i)$ for every $i, j \in W$. If $m \in f(j \circ i)$ then $v(\Diamond\Box jiA_m) = 1$ and so there is a substitution k such that $v(\Box kjiA_m) = 1$. In which case the substitution $k \circ j$ witnesses the truth of $v(\Diamond\Box iA_m) = 1$ so $m \in f(i)$. $\Box$

Let $X = \{1, ..., n\}$. For a given $Y \subseteq X$ we define, as before:

$D_Y := \bigwedge_{m \in Y} \Diamond\Box A_m \wedge \bigwedge_{m \notin Y} \neg\Diamond\Box A_m$

As before, D_Y and D_Z are inconsistent when $Y \neq Z$. Observe, also, that for any substitution i, $v(iD_{f(i)}) = 1$ since, by definition, $f(i)$ is just the set of m such that $v(\Diamond\Box iA_m) = 1$.

Suppose that $(P^0(X), \supseteq, V)$ is a Medvedev model, that $X = \{1, ..., n\}$ and that $A_1...A_n$ are a partition of propositional formulas such that $V(\{m\}, A_m) = 1$. We may define a substitution as follows:

$$i(p) = \bigvee_{V(Y,p)=1} D_Y$$

Proposition 12. *For any formula A and substitution j, $v(jiA) = V(f(j), A)$.*

Proof. We may prove this by induction of formula complexity.

Base case: Suppose that $V(f(j), p) = 1$. So $D_{f(j)}$ is a disjunct of $i(p)$, and since j distributes over disjunctions, $jD_{f(j)}$ is a disjunct of jip. By our observation $v(jD_{f(j)}) = 1$ so $v(jip) = 1$.

Conversely, suppose $v(jip) = 1$. So $v(jD_Y) = 1$ for some Y such that $V(Y, p) = 1$. By our observation we know that $v(jD_{f(j)}) = 1$. Since the D_Z's are pairwise incompatible (and, thus, so are their substitution instances) it follows that $v(jD_Y) = v(jD_{f(j)}) = 1$ only if $f(j) = Y$. So $V(f(j), p) = V(Y, p) = 1$

Inductive step: the negation and conjunction cases are straightforward, and the $\Box$ case follows from the fact that f is a p-morphism. Explicitly: if $v(\Box jiA) = 0$ then $v(kjiA) = 0$ for some substitution k. So $V(f(kj), A) = 0$ by the inductive hypothesis.

Conversely, if $V(f(j), \Box A) = 0$ then $V(Y, A) = 0$ for some non-empty $Y \subseteq f(j)$. By Lemma 10 there exists a k such that $f(k \circ j) = Y$, and so by the inductive hypothesis, $v(kjiA) = V(f(k \circ j), A) = V(Y, A) = 0$. $\Box$

Theorem 13. *If v is a valuation of the full substitution class, $\Delta_v \subseteq \mathsf{Med}$*

Proof. It suffices to show that any sentence C consistent in Med is consistent in Δ_v: there is some substitution k such that $v(kC) = 1$. If C is consistent in Med then there is a Medvedev model $(P^0(X), \supseteq, V)$ such that C is true at some world $Y \subseteq X$. Construct a substitution i as above, and pick some substitution j such that $f(j) = Y$ (this is possible by Lemma 10). Then $v(jiC) = V(f(j), C) = V(Y, C) = 1$. $j \circ i$ is the required substitution. $\Box$

3 The Logic of Logical Necessity

Which principles of modal logic are valid on the logical interpretation of $\Box$? The existence of an identity substitution immediately ensures the validity of the the T axiom, $\Box A \to A$. For given any substitution class S, and S-valuation, v, it follows that $v(\Box A) = 1$ only if $v(iA) = 1$ for any substitution and, in particular, that $v(\iota A) = v(A) = 1$ when ι is the identity substitution. Similarly, the closure of the substitutions under composition ensures the validity of the $\mathsf{S4}$ axiom, since if $v(\Box A) = 1$ then, for any pair of substitution $i, j \in S$, their composition $i \circ j$ is also in S, and so $v((i \circ j)A) = 1$. For fixed j, if $v(ijA) = 1$ for every $i \in S$, it follows that $v(\Box jA) = 1$; and since this holds for every $j \in S$, $v(\Box\Box A) = 1$. Indeed, McKinsey showed that for every substitution class S, $L(S)$ contains the theorems of $\mathsf{S4}$.[14]

Theorem 14 (McKinsey)**.** *Let S be any substitution class.*

1. *The set of validities $L(S)$ is closed under the rules of necessitation, modus ponens and uniform substitution.*

2. *The formulas $\Box A \to \Box\Box A$, $\Box A \to A$, $\Box(A \to B) \to (\Box A \to \Box B)$ are all valid.*

Thus every theorem of $\mathsf{S4}$ is valid with respect to the class of S-valuations.

Proof. This result appears in McKinsey (1945). We may obtain it directly from Proposition 2 by noting that the Kripke model associated with any S-valuation v is transitive and reflexive, as ensured, respectively, by the Composition and Identity conditions on substitution classes. It follows that for any A and B, $\Box A \to \Box\Box A$, $\Box A \to A$ and $\Box(A \to B) \to \Box A \to \Box B$ are true in any valuation. Closure under the rules of necessitation, modus ponens and uniform substitution follows straightforwardly from the definitions of validity and S-valuation. □

3.1 The Brouwer and McKinsey axioms

Which further principles of modal logic might be valid under the logical interpretation of necessity? Taking the logic of metaphysical necessity as our cue, one might wonder if the Brouwerian axiom

B $A \to \Box\Diamond A$

is valid. The result of adding B to $\mathsf{S4}$ yields $\mathsf{S5}$, a logic commonly supposed to be the logic of metaphysical necessity. In correspondence, McKinsey offers the following argument against the truth of the Brouwerian axiom.[15] While is certainly true that sugar is sweet, and vinegar is not, the Brouwerian axiom, if it were true, would further imply that it's necessarily possible that sugar is sweet and vinegar is not sweet. But given the left-to-right direction of the substitutional constraint, substituting sugar for vinegar, we may infer the absurd conclusion that it's possible that sugar is sweet and sugar is not sweet.

This conclusion is not peculiar to McKinsey's definition of logical necessity either, since it is also a consequence of our more general class of metalogical constraints on logical necessity. For as we noted there, so long as our account of logical truth, Δ, is closed under the rule of substitution we have the left-to-right direction of McKinsey's substitutional constraint. (The reader may have noticed that we actually *do* have a candidate notion of logical truth which isn't closed under the rule of substitution — pre-validity. We'll investigate this interpretation in section 3.2, where we will see that it is indeed consistent with the logic of $\mathsf{S5}$.)

What is special about the substitution of 'sugar' for 'vinegar' in 'sugar is sweet and vinegar is not' is that no further substitutions to the resulting sentence can change its truth value. Let us sharpen this notion:

[14] Drake (1962) improves on this result by showing that the intersection of the logics $L(S)$ for every substitution class S is exactly $\mathsf{S4}$.

[15] See Anderson and Belnap (1975, pp.122-123).

Definition 8. *A substitution $i \in S$ is terminal with respect to S if and only if $v(\Box iA) = 1$ or $v(\Box\neg iA) = 1$ for every sentence A of $\mathcal{L}$ and S-valuation v. A terminal substitution, i,* matches *to a $\top\bot$-substitution j if and only if $v(\Box ip) = 1$ when $jp = \top$ and $v(\Box\neg ip) = 1$ when $jp = \bot$.*

Finally, say that a substitution class contains all terminal substitutions when it has a terminal substitution matching any $\top\bot$ substitution, where, as before a $\top\bot$-substitution is a substitution whose range on $\mathcal{L}()$ is $\{\top, \bot\}$.

Clearly any substitution class containing a $\top\bot$ substitution, such as $S(\top\bot...)$ or H (the class of substitutions that leave letters alone or replace them with $\top$ or $\bot$) has a terminal substitution. However, many other formulas can play the role of $\top$, such as $(p_0 \vee \neg p_0)$ or $\Box p_0 \to \Box\Box p_0$, and similarly many formulas can play the role of $\bot$. So one need not have $\top$ and $\bot$ in the range of substitutions in order to have terminal substitutions.

The presence of terminal substitutions imposes its own distinctive modal logic. The McKinsey axiom says:

M $\Box\Diamond A \to \Diamond\Box A$

It characterises the Kripke frames in which each world sees a terminal world: a world that sees only itself.[16]

Proposition 15 (McKinsey)**.** *Let S be a substitution class containing a terminal substitution. Then the theorems of* S4M *are valid over the class of S-valuations.*

Proof. Since, by Theorem 14, the validities of any substititution class are closed under modus ponens and necessitation and contain the axioms of S4, it remains only to show that the McKinsey axiom M is valid. If $\Box\Diamond A$ is true in some arbitrary S-valuation v, then for some terminal substitution $i \in S$, $v(\Diamond iA) = 1$. Since i is terminal, $v(\Box iA) = 1$, and so $v(\Diamond\Box A) = 1$. □

The McKinsey axiom is compatible with Brouwer's axiom. However, the only consistent modal logic containing S4, McKinsey's axiom, and Brouwer's axiom is the trivial modal logic Triv in which $\Box$ just means *true*.[17] The modal logic Triv has as its characteristic axiom:

Triv $\Box A \leftrightarrow A$

Triv is the logic of the trivial substitution class that has the identity substitution ι as its sole element.

Indeed, according to the Brouwerian axiom, truths of the form $\Box\Diamond A$ are quite commonplace: for any sentence A, either $\Box\Diamond A$ or $\Box\Diamond\neg A$ is true, depending on whether A or $\neg A$

[16] McKinsey actually considers the formula:

F $\Box\Diamond A \wedge \Box\Diamond B \to \Diamond(A \wedge B)$

which is only equivalent to M in the context of S4. The principle M appears to originate from Cresswell and Hughes (1996).

[17]

1. $A \to \Box\Diamond A$ (an instance of B)
2. $\Box\Diamond A \to \Diamond\Box A$ (an instance of M)
3. $\Diamond\Box A \to \Box A$ (an equivalent of the 5 axiom)
4. $A \to \Box A$ from 1-3
5. $\Box A \to A$ by T.

is true. By contrast, in a substitution class containing all terminal substitutions, truths of the form $\Box\Diamond A$ are quite special: it may be shown that the sentence $\Box\Diamond A$ is true *only if* A is a theorem of Triv, or equivalently, if the result of deleting all modal operators from A is a tautology. Let us write A^- for the propositional formula that results from deleting all modal operators from A. Then:

Proposition 16. *Suppose S is a substitution class for which there is at least one S-valuation and which contains all terminal substitutions. Then the following are equivalent.*

1. *$\Box\Diamond A$ is true in all S-valuations.*
2. *$\Box\Diamond A$ is true in some S-valuation.*
3. *A^- is a tautology.*
4. *A is a theorem of* Triv.

Proof. The equivalence of 3 and 4 is straightforward. Clearly 1 implies 2. We show that 2 implies 3 and 3 implies 1, establishing the equivalence of 1-3.

We may show by a straightforward induction that if i is a terminal substitution, then $v(iB) = v(i(B^-))$ for any B. The only non-trivial case is to show that $v(i\Box C) = v(i(\Box C)^-)$. $v(i\Box C) = v(\Box iC) = 1$ iff $v(iC) = 1$ since i is terminal. Moreover, by the inductive hypothesis, $v(iC) = 1$ iff $v(i(C^-)) = 1$. But $v(i(\Box C)^-) = v(i(C^-))$. So $v(i\Box C) = v(i((\Box C)^-))$.

Assume 2, and let v be an S-valuation such that $v(\Box\Diamond A) = 1$. Thus for every terminal substitution i, $v(\Diamond iA) = v(iA) = 1$. So $v(iA^-) = 1$ for every terminal substitution i, which clearly implies that A^- is a tautology. This establishes 3.

Now suppose that A^- is a tautology and j an arbitrary S substitution. Let i be any terminal substitution and v any S-valuation. $v(ijA) = v(i((jA)^-))$ from before. But if A^- is a tautology, then $(jA)^-$ is a substitution instance of A^- of the letters by sentences of propositional logic (specifically the substitution j^- where $j^-(p_k) = (jp_k)^-$). So $(jA)^-$ is a tautology too, and hence $v(ijA) = v(i((jA)^-) = 1$. Thus i witnesses the truth of $v(\Diamond jA) = 1$; and since j was an arbitrary S substitution, $v(\Box\Diamond A) = 1$ $\Box$

3.2 Carnap's Theory of Logical Necessity

Earlier we reported a general argument, due to McKinsey, that the logic of logical necessity does not contain the Brouwerian axiom. It rested on the proposed constraint connecting logical necessity and logical truth, along with the assumption that logical truth is closed under the rule of substitution. Might we resist this argument by adopting a conception of logic that is not closed under the rule of substitution? Interestingly, an early example of a theory of logical necessity in Carnap (1946) and Carnap (1947, pp.173-177) results in a notion of logical truth that fails to be closed under the rule of substitution (see Cresswell (2013)), whilst simultaneously validating all the theorems of $\mathsf{S5}$, including all instances of Brouwer's axiom.

Indeed, we have encountered our own candidate notion of logical truth not closed under substitution: pre-validity. This raises the question of whether there might be a substitutional interpretation of $\Box$ as pre-validity:

> Is there a substitution class S such that, for any S-valuation v, $v(\Box A) = 1$ if and only if A is pre-valid with respect to S?

Indeed, we will show that there is and that the pre-validities of this class coincide with Carnap's notion of logical truth for propositional modal logic.[18]

Recall from Example 5 in section 1.1 that K is the class of substitutions i such that $i(p_k)$ is either p_k itself, or the result of prefixing a finite string of negations in front of p_k. Note that there is in effect no difference between substitutions with the same parity: if for each k, ip_k and jp_k agree on whether they contain an even or odd number of negations, then for the purposes of valuation we might as well treat these substitutions as the same. However this redundancy is necessary to meet the formal requirement that a substitution class be closed under substitution.

Proposition 17. *For any K-valuation v, $v(\Box A) = 1$ iff A is pre-valid with respect to K.*

Proof. Given any K-valuation v and substitution $i \in K$, let (iv) be the unique K-valuation defined by setting $(iv)(p_k) = v(ip_k)$ (so $(iv)(p_k) = 1$ iff ip_k has an even number of negations and $v(p_k) = 1$, or ip_k has an odd number of negations and $v(p_k) = 0$).

We show by induction on the modal degree of A that for any v and $i \in K$:

$$v(iA) = (iv)(A)$$

The identity holds of the sentence letters by construction and is clearly inherited over truth-functional compounds. So suppose that for sentences of modal degree n, $v(iA) = (iv)(A)$ for any K-valuation v and substitution $i \in S$. Recall that $v(\Box iA) = 1$ iff $v(jiA) = 1$ for every substitution $j \in K$. Now for any j, we may apply the inductive hypothesis to v and the substitution $j \circ i$ to get $v(jiA) = (jiv)(A)$; and by similarly applying the inductive hypothesis to iv and j, we get that $(jiv)(A) = (iv)(jA)$. So $v(jiA) = 1$ for every $j \in S$ iff $(iv)(jA) = 1$ for every $j \in S$, which holds iff $(iv)(\Box A) = 1$, as required.

We may now prove the proposition. Note that for a fixed v, every K-valuation u is of the form iv for some substitution $i \in K$. (Set $i(p_k) = p_k$ if $u(p_k) = v(p_k)$, and let $i(p_k) = \neg p_k$ if $u(p_k) \neq v(p_k)$.) So $v(\Box A) = 1$ iff $v(iA) = 1$ for every $i \in K$, iff $(iv)(A) = 1$ for every $i \in K$, iff $u(A) = 1$ for every K-valuation, iff A is pre-valid with respect to K. $\Box$

It is also easy to verify that all the theorems of S5 are valid with respect the class of K valuations.

We now compare the pre-logic of our substitution class with the propositional modal logic of Carnap's theory of logical necessity.[19] Formulas of propositional modal logic are evaluated with respect to truth-value assignments to sentence letters, $v^- : \mathcal{L}() \to \{0, 1\}$. We define what it means for a sentence A of propositional *modal* logic, $\mathcal{L}(\wedge\neg\Box)$, to be true at a truth-value assignment v^-, written $v^- \models A$, as follows:

- $v^- \models p_k$ iff $v^-(p_k) = 1$
- $v^- \models A \wedge B$ iff $v^- \models A$ and $v^- \models B$
- $v^- \models \neg A$ iff $v^- \not\models A$
- $v^- \models \Box A$ iff $u^- \models A$ for every truth-value assignment u^-.

[18] See Carnap (1946). The propositional fragment of Carnap's logic has been treated in Cresswell (2013), Thomason (1973), Hendry and Pokriefka (1985). A related project is carried out in Cocchiarella (1974); see also Carroll (1978).

[19] A presentation of the propositional fragment of Carnap's theory may be found in Cresswell (2013).

Thus $\Box A$ can be taken to mean in an extended sense that A is a tautology. A sentence A is C-*valid* iff $v^- \models A$ for every truth-value assignment v^-. C-validity is not closed under the rule of substitution: for instance, $\neg\Box p_0$ is C-valid, while $\neg\Box\neg(p_0 \wedge \neg p_0)$ is not. Much like our Proposition 17, a sentence of the form $\Box A$ is true at a propositional valuation iff A is C-valid. Indeed, we can show:

Proposition 18. *A is C-valid iff A is pre-valid with respect to K.*

Proof. Given a K-valuation v, we write v^- for corresponding the truth-value assignment: $v \restriction_{\mathcal{L}()}$. By Proposition 3 every truth-value assignment is identical to v^- for some unique K-valuation v, and so we lose no generality by restricting attention to truth-value assignments of the form v^-. We shall show by induction that for every K-valuation v , $v(A) = 1$ iff $v^- \models A$.

The identity clearly holds for sentence letters. The clauses for the truth functional connectives are straightforward. $v^- \models \Box A$ iff $u^- \models A$ for every truth-value assignment u^-. So by the inductive hypothesis, $u(A) = 1$ for every K-valuation u. Now for every substitution $i \in K$, iv (as defined in Proposition 17) is a K-valuation, so $(iv)(A) = 1$ and since $(iv)(A) = v(iA)$, $v(iA) = 1$ for every $i \in K$. Thus $v(\Box A) = 1$. Conversely, since every K-valuation is of the form iv for some $i \in K$, if $v(\Box A) = 1$, then $u(A) = 1$ for every valuation, so by the inductive hypothesis $u^- \models A$ for every u and so $v^- \models \Box A$.

As we saw, A is pre-valid iff $v(\Box A) = 1$. By the above, $v(\Box A) = 1$ iff $v^- \models \Box A$ iff A is C-valid. This completes the proof. $\Box$

4 The Logic of Specific Substitution Classes

The foregoing remarks give us some idea of what sort of logical principles must be a part of the logic of logical necessity. The situation becomes more intricate when we look at specific substitution classes. We shall see that the logics of some, though not all, substitution classes include the Grzegorczyk axiom Grz

Grz $\Box(\Box(A \to \Box A) \to A) \to A$

a principle that is not a theorem of S4M.

We shall also see that the logics of some, though not all, substitution classes include an axiom we call "the subset principle" that is distinctive of Med.

4.1 The logic of non-modal substitution classes

In this section we investigate various classes of of non-modal substitutions — substitutions that map letters to non-modal propositional formulas. In his book *Philosophical Applications of Modal Logic*, Lloyd Humberstone provides an interpretation of McKinsey's theory of modality in terms of (what we have called) the class of Humberstone substitutions, H. These are substitutions that either map a letter to itself, $\top$ or $\bot$. After noting that every theorem of S4M is valid with respect to this substitution class (Proposition 14), he poses the following questions (p.168):

1. Are the theorems of S4M exactly the validities with respect to the class H?

2. How sensitive is the logic to the exact substitution class used? Is the logic of H the same as the logic of $S(\top\bot)$ or the logic of the full substitution class?

The first question may be answered negatively. In particular, the Grzegorczyk axiom Grz above is valid with respect to the class of H-valuations but is not a theorem of $\mathsf{S4M}$.[20]

Proposition 19. Grz *is valid with respect to H-valuations.*

Proof. Let v be an arbitrary H-valuation and A an arbitrary formula. We suppose that $v(A) = 0$ and then show that $v(\Box(\Box(A \to \Box A) \to A)) = 0$. Our goal is thus to find a substitution i such that $v(iA) = 0$ and $v(\Box(iA \to \Box iA)) = 1$.

Without loss of generality, we may restrict our attention to substitutions that map sentence letters not appearing in A to $\top$. Partially order these remaining substitutions as follows: $i \leq j$ when there exists a k such that $j = k \circ i$ (or, equivalently, for all n, $j(p_n) = \top$ whenever $i(p_n) = \top$ and $j(p_n) = \bot$ whenever $i(p_n) = \bot$).

Because there are only finitely many letters in A, there are only finitely many substitutions in this ordering. Since $v(A) = 0$, there must be a maximal substitution, i, with respect to this ordering such that $v(iA) = 0$. This means that $v(iA) = 0$ and that for any $j > i$ $v(jA) = 1$. We now show that $v(\Box(iA \to \Box iA) = 1$. For any j, either (a) $j \circ i = i$, so $v(jiA) = 0$ or else (b) $j \circ i > i$, in which case $v(\Box jiA) = 1$, for $v(kjiA) = 1$ for any $k \in H$, since i was the maximal substitution making A false and $k \circ j \circ i \geq j \circ i > i$. Either way, $v(jiA \to \Box jiA) = 1$ and since j was arbitrary, $v(\Box(iA \to \Box iA) = 1$. This completes the argument. $\Box$

We can also use Grz to make progress with Humberstone's second question regarding $S(\top\bot)$. Recall that the difference between $S(\top\bot)$ and H is that while H substitutions must map p to itself or to $\top$ or $\bot$, an $S(\top\bot)$ substitution can map p to any other letter, in addition to $\top$ and $\bot$. We can now show:

Proposition 20. $\mathsf{Grz} \notin L(S(\top\bot))$ *and, specifically,* Grz *may be invalidated through the instance $A = p \to \Box p$.*

Proof. Let v be a $S(\top\bot)$-valuation with a true and a false letter, p and q. We will show that $v(\Box(\Box(A \to \Box A) \to A)) = 1$ and $v(A) = 0$ when $A = p \to \Box p$. Let i be any substitution in $S(\top\bot)$, and suppose $v(i(p \to \Box p)) = 0$. It straightforwardly follows that ip cannot be $\top$ or $\bot$, and so it must be some sentence letter r. Let j be a substitution such that $jr = q$, the false sentence letter.

Then $v(jiA) = v(q \to \Box q) = 1$ since q is false, and $v(ji\Box A) = v(\Box(q \to \Box q)) = 0$ since $p \to \Box p$ is a substitution instance of $q \to \Box q$ and is false. Thus $v(jiA \to \Box jiA) = 0$, and so $v(\Box(iA \to \Box iA)) = 0$. So we have shown that, for an arbitrary substitution $i \in S(\top\bot)$, when $v(iA) = 0$ then $v(\Box(iA \to \Box iA)) = 0$ and hence that $v(\Box(\Box(A \to \Box A) \to A)) = 1$. Finally, $v(p \to \Box p) = 0$ since p is true and $\Box p$ false in v. $\Box$

We can extend the above line of argument to show that Grz is invalid with respect to the class of all non-modal substitutions $S(\neg\wedge)$. In order to do this, we need to appeal to the following fact about propositional logic:

Lemma 21. *Suppose that A is a sentence of the propositional calculus that is neither tautologous nor contradictory, and that B is any other sentence (possibly involving modal operators). Then there exists a substitution $i \in S(\neg \wedge \Box)$ such that iA is equivalent to B.*

[20] To see that Grz is not a theorem of $\mathsf{S4M}$, consider the frame $(\{0,1,2\}, \{(0,1),(1,0),(0,2),(1,2)\} \cup \{(w,w) \mid w = 0,1,2\})$ and the model over it in which p is true at 0 and 2, but not at 1. Since it is transitive, reflexive and every world sees a terminal world, the theorems of $\mathsf{S4M}$ hold in this model, but $\Box(\Box(p \to \Box p) \to p)$ holds at 1 while p does not.

When $B \in \mathcal{L}(\neg\wedge)$, this substitution will be $S(\neg\wedge)$. When $B \in \mathcal{L}(\top\bot\neg)$, there will similarly exist a substitution $i \in S(\top\bot\neg)$ such that iA is equivalent to B.

Proof. Let $p_1...p_n$ be the sentence letters in A and consider the truth-value assignments to $p_1...p_n$. Since A is neither tautologous nor contradictory, there are assignments v and v' making A true and false respectively. Since v' may be obtained from v by picking a letter p_k from $p_1...p_n$ and flipping its truth value, and repeating this as many times as necessary, there must exist some assignment u and letter p_k such that u makes A true, but the assignment that results from flipping the truth value of p_k in u makes A false. We can then define the desired substitution as follows:

$$ip_m = \begin{cases} B & \text{if } m = k \text{ and } u(p_k) = 1 \\ \neg B & \text{if } m = k \text{ and } u(p_k) = 0 \\ \neg(p_0 \wedge \neg p_0) & \text{if } m \neq k \text{ and } u(p_k) = 1 \\ p_0 \wedge \neg p_0 & \text{if } m \neq k \text{ and } u(p_k) = 0 \end{cases}$$

Clearly $i \in S(\neg\wedge)$ when $B \in \mathcal{L}(\neg\wedge)$

For $B \in \mathcal{L}(\top\bot\neg)$ we may analogously obtain a substitution $i \in S(\top\bot\neg)$ such that iA is equivalent to B by replacing $\neg(p_0 \wedge \neg p_0)$ with $\top$ and $p_0 \wedge \neg p_0$ with $\bot$ in the above definition. $\square$

Proposition 22. $\mathsf{Grz} \notin L(S(\neg\wedge))$ *and* $\mathsf{Grz} \notin L(S(\top\bot\neg))$.

Proof. The proof is completely parallel to the proof of Proposition 20, except that when we assume that i is a $S(\neg\wedge)$ or $S(\top\bot\neg)$ substitution and that $v(i(p \to \Box p)) = 0$ we may infer that i is neither tautologous nor contradictory and can apply Lemma 21 to obtain the relevant substitution j. $\square$

We end our discussion of these non-modal substitution classes by relating them to Kripke semantics. For the substitution class H we will work with the class of partial function frames $(W, \subseteq)$ where:

> $W := X \rightharpoonup \{\top, \bot\}$ (the set of partial functions from a finite set X to a two valued set).

We call the logic of these frames *the logic of finite partial functions*.

Recall that the Kripke frame (W_H, R_H) associated with H is defined by letting $W_H = H$ and $R_H = \{(i, j \circ i) \mid i, j \in H\}$, and the valuation V on (W_H, R_H) associated with a H-valuation v is defined by $V(i, p_k) = v(ip_k)$..

Proposition 23. *For any finite set X there is a surjective p-morphism, f, from the Kripke frame associated with H, (W_H, R_H), to the finite partial function frame $(X \rightharpoonup \{\top, \bot\}, \subseteq)$.*

Moreover, for any H-valuation v and letters $p_1...p_n$ where $n = |X|$, there exists a valuation U for which the truth-values of $p_1...p_n$ are preserved by f:

> $V(i, p_k) = U(f(i), p_k)$ *for* $k = 1...n$

here V is the valuation associated with v, defined by $V(i, p_k) = v(ip_k)$.

Proof. For convenience we will let $X = \{p_1 ... p_n\}$. Define a function $f : H \to (\{p_1...p_n\} \rightharpoonup \{\top, \bot\})$ as follows:

$$f(i)(p_k) = \begin{cases} \text{undefined} & \text{if } ip_k = p_k \\ ip_k & \text{otherwise} \end{cases}$$

It remains to show that f satisfies the two conditions for being a p-morphism. To establish the first condition we must show that $f(i) \subseteq f(j \circ i)$ for any i and j in H. If $f(i)(p_k)$ is defined and $= \top$ then $ip_k = \top$ and so $jip_k = \top$, and thus $f(j \circ i)(p_k) = \top$. Similarly if $f(i)(p_k)$ is defined and $= \bot$ then $f(j \circ i)(p_k) = \bot$, so $f(i) \subseteq f(j \circ i)$.

The second condition amounts to showing that, for any $i \in H$ and $w \in \{p_1...p_n\} \rightharpoonup \{\top, \bot\}$, if $f(i) \subseteq w$ then there exists a substitution $j \in H$ such that $f(j \circ i) = w$. $j(p_k)$ may be defined as $w(p_k)$ if $w(p_k)$ is defined, and as p_k otherwise.

Given a H-valuation v, we can now define a valuation U on $(\{p_1...p_k\} \rightharpoonup \{\top, \bot\})$:

$$U(w, p_k) = \begin{cases} v(p_k) & \text{if } w(p_k) \text{ is undefined} \\ 1 & \text{if } w(p_k) = \top \\ 0 & \text{if } w(p_k) = \bot \end{cases}$$

To establish that it respects the letters $p_1...p_n$ we must show that $v(ip_k) = V(f(i), p_k)$ for each $k = 1, ..., n$. If $ip_k = p_k$, then $f(i)(p_k)$ is undefined and so by the definition of V, $V(f(i), p_k) = v(p_k)$. But since $ip_k = p_k$, $V(f(i), p_k) = v(ip_k)$ as required. Otherwise $ip_k = \top$ or $ip_k = \bot$. In the former case $V(f(i), p_k) = 1$ by definition of V, and $v(ip_k) = v(\top) = 1$, so they are identical as required. The latter case is proved in the same manner.

□

Theorem 24. *$L(H)$ is the logic of finite partial functions.*

Proof. We shall show that a sentence is consistent with $L(H)$ iff it is satisfiable in some finite partial function frame.

If A is consistent with $L(H)$ then for some (arbitrary) substitution instance A' and some H-valuation, v, $v(A') = 1$. Let $p_1...p_n$ be the letters in A'. It is immediate from Proposition 23 that there is a valuation U over the partial function frame $(\{p_1...p_n\} \rightharpoonup \{\top, \bot\}, \subseteq)$ such that $U(f(\iota), A') = 1$ and so A' is satisfiable in a finite partial function frame. A thus must be also consistent in the logical of partial function frames, for otherwise its substitution instances would be inconsistent too.

Now suppose that A is true at some world w in a partial function model $(X \rightharpoonup \{\top, \bot\}, \subseteq, U)$: $U(w, A) = 1$. By Proposition 23 there exists a surjective p-morphism from (W_H, R_H) to $(X \rightharpoonup \{\top, \bot\}, \subseteq)$. Choose some substitution $k \in H$ such that $f(k) = w$.

Here we follow the proof of Proposition 12. Let v be any H-valuation and V the associated valuation on (W_H, R_H). We can construct a substitution, i (which need not necessarily belong to H), such that for any $j \in H$, $V(j, ip_m) = U(f(j), p_m)$. Then, because f is a p-morphism, $V(j, iB) = U(f(j), B)$ for any sentence B (see Proposition 12). So in particular, $V(k, iA) = U(f(k), A) = U(w, A) = 1$. Thus some substitution instance of A, namely kiA, is true in a H-valuation, namely v, and thus A is consistent in $L(H)$.

Here is how we find such a substitution i. For each $w : X \rightharpoonup \{\top, \bot\}$ define a sentence

$$C_w := \bigwedge_{w(p_m)=\top} \Box p_m \wedge \bigwedge_{w(p_m)=\bot} \Box\neg p_m \wedge \bigwedge_{p_m \notin dom(w)} (\Diamond p_m \wedge \Diamond\neg p_m)$$

where p_m ranges over the letters $p_1...p_n$. Now consider the substitution:

$$ip_m := \bigvee_{U(w,p_m)=1} C_w$$

Observe firstly that C_w and C'_w are inconsistent in the propositional calculus when $w \neq w'$.

First we establish that for any $j \in H$, $V(j, C_{f(j)}) = v(jC_{f(j)}) = 1$. There is a conjunct of $jC_{f(j)}$ for each p_m where $m \in \{1, ..., n\}$: we will show that each such conjunct is true. If $f(j)(p_m)$ is undefined, then $jp_m = p_m$ and the relevant conjunct $\Diamond jp_m \wedge \Diamond\neg jp_m$ is true according to v. If $f(j)(p_m) = \top$ or $\bot$ then $jp_m = \top$ or $\bot$, respectively, and the relevant conjunct, $\Box jp_m$ or $\Box\neg jp_m$ respectively, is true according to v.

Now $V(j, ip_m) = 1$ iff $v(jip_m) = v(j \bigvee_{U(w,p_m)=1} C_w) = 1$ iff for some w such that $U(w, p_m) = 1$, $v(jC_w) = 1$. $v(jC_{f(j)} = 1$ and the Cs are pairwise incompatible, $w = f(j)$. So the last statement holds iff $U(f(j, p_m) = 1$ as required.

$\square$

Next we turn to the logic of the non-modal substitution class. Let C denote the set of substitutions of letters for arbitrary non-modal sentences (i.e. $S(\neg\wedge)$). Consider the class of frames $\mathcal{F}_X = (W, \leq)$, where X is a finite set and:

$W := \{(P, a) \mid P \subseteq \{0, 1\}^X, a \in P\}$

$(P, a) \leq (Q, b)$ iff $P \supseteq Q$.

The logic of the substitution class C may similarly be related to the logic of this class of frames.

Proposition 25. *For any finite set X there is a surjective p-morphism, f, from the Kripke frame associated with C, (W_C, R_C), to the frame $\mathcal{F}_X$.*

Moreover, for any C-valuation v and letters $p_1...p_n$ where $n = |X|$, there exists a valuation U for which the truth-values of $p_1...p_n$ are preserved by f:

$V(i, p_k) = U(f(i), p_k)$ *for* $k = 1...n$

Proof. Let v be any C-valuation. Without loss of generality we will suppose that $X = \{p_1...p_n\}$. Given a finite set of letters, Z, and $a \in 2^Z$, let B^a be the formula $\bigwedge_{a(p_m)=1} p_m \wedge \bigwedge_{a(p_m)=0} \neg p_m$.

Define a function $f : C \to \{(P, a) \mid P \subseteq 2^X, a \in P\}$ as follows:

$$f(i) = (\{a \in 2^X \mid v(\Diamond iB^a) = 1\}, v \circ i \upharpoonright X)$$

It remains to show that f satisfies the two conditions for being a p-morphism. To establish the first condition we must show that $f(i) \leq f(j \circ i)$ for any i and j in C. Suppose a belongs to the first component of $f(j \circ i)$, so that $v(\Diamond jiB_a) = 1$. So for some $k \in C$, $v(kjiB^a) = 1$. $k \circ j \in C$, so $v(\Diamond iB^a) = 1$, which means that a belongs to the first component of $f(i)$. Since a was arbitrary, we have shown that $f(i) \leq f(j \circ i)$.

Now suppose that $f(i) \leq (Q, e)$, where $f(i) = (P, d)$. We wish to find a substitution j such that $f(j \circ i) = (Q, b)$. Let $q_1...q_r$ be the letters appearing in $ip_1...ip_n$, and call members of $2^{\{q_1...q_r\}}$ truth-value assignments to $q_1...q_r$. Given $a \in 2^{\{q_1...q_r\}}$ and a propositional formula A in the letters $q_1...q_r$ we write $a(A)$ for As truth-value under the assignment a. Let $a \circ i : \{p_1...p_n\} \to \{0, 1\}$ be the function $p_m \mapsto a(ip_m)$, and let Y be the set

$\{a \in 2^{\{q_1...q_r\}} \mid a \circ i \in Q\}$. (Note that *every* element of P is of the form $a \circ i$ for some $a \in X$.) For any $a \in 2^{\{q_1...q_r\}}$ there is a corresponding $\top\bot$ substitution, k_a, defined on $\{q_1...q_r\}$ defined by $k_a(q_m) = \top$ if $a(q_m) = 1$ and $k_a(q_m) = \bot$ otherwise. For any propositional formula A in the letters $q_1...q_r$ it is clear that $v(k_a A) = a(A)$.

Given a surjection $\sigma : 2^{\{q_1...q_r\}} \to Y$, we can define a substitution on $q_1...q_r$ (as in Lemma 10):

$$j(q_m) = \bigvee_{\sigma(a)(q_m)=1} B^a$$

where a ranges over members of $2^{\{q_1...q_r\}}$. Notice that $\sigma(a)(p_m) = 1$ iff B^a is a disjunct of $j(q_m)$, iff $a(jq_m) = 1$, iff $v(k_a jq_m) = 1$. So we have $\sigma(a)(q_m) = a(jq_m) = v(k_a jq_m)$, for any $a \in 2^{\{q_1...q_r\}}$ and $m \in \{1, ..., r\}$, and so for any propositional formula A in letters $q_1...q_r$:

$$\sigma(a)(A) = a(jA) = v(k_a jA)$$

Since iB^c is a propositional formula, we get that$\sigma(a)(iB^c) = a(jiB^c) = v(k_a jiB^c)$.

We firstly show that $\{a \mid v(\Diamond jiB^a) = 1\} = Q$. If $c \in Q$ then $c = b \circ i$ for some $b \in Y$ and so there exists an a such that $\sigma(a) = b$ since σ is surjective. So $v(k_a jiB^c) = \sigma(a)(iB^c) = b(iB^c) = c(B^c) = 1$. So for any $c \in Q$, $v(\Diamond jiB^c) = 1$, and thus $Q \subseteq \{c \in 2^X \mid v(\Diamond jiB^c) = 1\}$. Conversely, if $v(\Diamond jiB^c) = 1$ then for some substitution k, $v(kjiB^c) = 1$. Indeed, since jiB^c is a propositional formula, we may assume without loss of generality that k is a $\top\bot$-substitution, k_a for some $a \in X$. So $1 = v(k_a jiB^c) = \sigma(a)(iB^c) = (\sigma(a) \circ i)(B^c)$, and since $\sigma(a) \in Y$, $\sigma(a) \circ i \in Q$. But $(\sigma(a) \circ i)(B^c) = 1$ iff $\sigma(a) \circ i = c$, $c \in Q$.

We would further like $v(jiB^e) = 1$, so that $f(j \circ i) = (Q, e)$ as required of a p-morphism. Since $e \in Q$, we know there exists a substitution k such that $v(kjiB^e) = 1$. Moreover, since jiB^e is a propositional formula, any substitution k' such that $v(kq_m) = v(k'q_m)$ will also be such that $v(k'jiB^e) = 1$. So we may assume without loss of generality that k is a substitution that maps q_m to itself or $\neg q_m$ for $m = 1...r$. We claim that $k \circ j$ is the required substitution. By construction, $v(kjiB^e) = 1$, but also, since we have replaced literals for literals, $\{a \mid v(\Diamond kjiB^a) = 1\} = \{a \mid v(\Diamond jiB^a) = 1\} = Q$.

Now define a valuation U on $\mathcal{F}_X$. For $m = 1...n$:

$$U((P, d), p_m) = d(p_m)$$

U may be set arbitrarily on the remaining letters. It is immediate f preserves the truth-values of $p_1...p_n$: $f(i) = (P, v \circ i)$ for some P, so $V(i, p_m) = v(ip_m) = v \circ i(p_m) = U((P, v \circ i), p_m)$.

□

Theorem 26. *$L(C)$ is the logic of the frames $\mathcal{F}_X$ for finite sets X.*

Proof. We shall show that a sentence is consistent with $L(C)$ iff it is satisfiable in some frame $\mathcal{F}_X$.

If A is consistent with $L(C)$ then for some (arbitrary) substitution instance A' and some C-valuation, v, $v(A') = 1$. Let $X = \{p_1...p_n\}$ be the set of letters in A'. It is immediate from Proposition 25 that there is a valuation U over the frame $\mathcal{F}_X$ such that $U(f(\iota), A') = 1$ and so A' is satisfiable in $\mathcal{F}_X$. A must be also consistent in the logical of partial function frames, for otherwise its substitution instances would be inconsistent too.

Now suppose that A is true at some world (P, d) in a model $(\mathcal{F}_X, U)$: $U((P, d), A) = 1$. By Proposition 25 there exists a surjective p-morphism from (W_C, R_C) to $\mathcal{F}_X$. Choose some substitution $k \in C$ such that $f(k) = (P, d)$.

Here we follow the proof of Proposition 12. Let v be any C-valuation and V the associated valuation on (W_C, R_C). We can construct a substitution, i (which need not necessarily belong to C), such that for any $j \in H$, $V(j, ip_m) = U(f(j), p_m)$. Then, because f is a p-morphism, $V(j, iB) = U(f(j), B)$ for any sentence B in letters $p_1 ... p_n$. So in particular, $V(k, iA) = U(f(k), A) = U((P, d), A) = 1$. Thus some substitution instance of A, namely kiA, is true in a C-valuation, namely v, and thus A is consistent in $L(C)$.

For each world (P, d) of $\mathcal{F}_X$ define a sentence:

$$C_{P,d} := \bigwedge_{a \in P} \Diamond B^a \wedge \bigwedge_{a \in 2^X \setminus P} \neg \Diamond B^a \wedge B^d$$

Where, as before, $B^a = \bigwedge_{a(p_m)=1} p_m \wedge \bigwedge_{a(p_m)=0} \neg p_m$ and m ranges from 1 to n. Now consider the substitution:

$$ip_m := \bigvee_{U((P,d),p_m)=1} C_{P,d}$$

Observe firstly that $C_{P,d}$ and $C_{P',d'}$ are inconsistent in the propositional calculus when $(P, d) \neq (P', d')$.

First it can be shown that for any $j \in C$, $V(j, C_{f(j)}) = v(jC_{f(j)}) = 1$. In fact, this is trivial from the definition of f: the conjuncts of the form $\Diamond jB^a$ in $C_{f(j)}$ are defined as those where $v(\Diamond jB^a) = 1$, the conjuncts of the form $\neg \Diamond jB^a$ in $C_{f(j)}$ are defined as those where $v(\Diamond jB^a) \neq 1$, and the final conjunct is $B^{v \circ j}$ and $v \circ j(B^{v \circ j}) = 1$ (since $a(B^a) = 1$ for any $a \in 2^X$).

Now $V(j, ip_m) = 1$ iff $v(jip_m) = v(j \bigvee_{U((P,d),p_m)=1} C_{P,d}) = 1$ iff for some (P, d) such that $U((P, d), p_m) = 1$, $v(jC_{P,d}) = 1$. $v(jC_{f(j)} = 1$ and since the Cs are pairwise incompatible, $(P, d) = f(j)$. So the last statement holds iff $U(f(j), p_m) = 1$ as required.

□

The final non-modal substitution class we will consider is K. In section 3.2 we showed that the pre-validities of K were identical to the C-valid sentences — the sentences valid according to Carnap's interpretation of propositional modal logic. In Thomason (1973) an axiom system extending $\mathsf{S5}$ is presented, and it is shown to be complete with respect to the C-valid sentences. It consists of the result of closing all instances of the following axioms under modus ponens and the rule of necessitation (but *not* the rule of uniform substitution):[21]

PC Any instance of a propositional tautology.

K $\Box(A \to B) \to \Box A \to \Box B$

T $\Box A \to A$

5 $\neg \Box A \to \Box \neg \Box A$

Log $\neg \Box A$ when A is a propositional formula that is not tautologous.

It follows from Proposition 17 and Thomason's completeness result that this system completely axiomatizes the pre-validities of K.

[21] This system is closely related to $\mathsf{S13}$ from Cocchiarella (1974), the main difference being that Cocchiarella's system doesn't have a primitive notion of negation.

What of the *validities* of the substitution class K, $L(K)$? In light of the fact that C-validity is not closed under the rule of substitution, Cresswell (2013) has proposed that we treat a sentence as valid according to the Carnapian interpretation of $\Box$ only if all of its substitution instances are C-valid. Thus Cresswell's proposed notion of validity stands to Carnap's as the present notion of validity with respect to a substitution class stands to pre-validity (as in definitions 2 and 3). Cresswell proves that the validities of Carnap's theory are exactly the theorems of $\mathsf{S5}$. Thus we have

Corollary 27. $L(K) = \mathsf{S5}$

Proof. From Proposition 17 and Theorem 3.3 of Cresswell (2013). $\Box$

4.2 The logic of the full substitution class

Section 2.1 established the existence of a valuation for the full substitution class $S(\neg \wedge \Box)$, and so we know that the logic of this class is non-trivial. While we do not know what the logic of the full substitution class is, we will show in this section that, given a variant of Friedman's conjecture (Conjecture 4 above), it is exactly Med, and that even if this conjecture is false, the logic of the full substitution class contains Sub, a distinctive principle belonging to Med.

The uniqueness conjecture tells us that for any $S(\neg \wedge \Box)$-valuation v, $v(\Box A) = 1$ iff $A \in \mathsf{Med}$. We may now see how this conjecture settles the logic of the full substitution class:

Proposition 28. *Given the uniqueness conjecture,* $L(S(\neg \wedge \Box)) = \mathsf{Med}$.

Proof. Suppose $A \in \mathsf{Med}$. So for any $S(\neg \wedge \Box)$-valuation v, $v(\Box A) = 1$ by the conjecture and Theorem 9. So $v(iA) = 1$ for every substitution $i \in S(\neg \wedge \Box)$. Since v was arbitrary, A is valid. Conversely, suppose that A is valid, so that $v(iA) = 1$ for every substitution $i \in S(\neg \wedge \Box)$ and $S(\neg \wedge \Box)$-valuation v. By Theorem 9 we know there is at least one such v, and by the condition for necessity, we know $v(\Box A) = 1$. Moreover, according to the valuation constructed in Theorem 9 $v(\Box A) = 1$ only if $A \in \mathsf{Med}$. $\Box$

Note that we only appealed to the conjecture in proving one of the two directions, allowing us to establish the following without assuming the conjecture.

Proposition 29. $\mathsf{S4M} \subseteq L(S(\neg \vee \Box)) \subseteq \mathsf{Med}$.

This rules out some extensions of $\mathsf{S4M}$, but obviously leaves open any modal logic between $\mathsf{S4M}$ and Med.

One distinctive feature of Med is that it contains something we will call the "subset principle". To state this principle we will begin by describing some formulas D_Y for $Y \in P^0(X)$ that characterize the worlds of certain sorts of Medvedev models of the form $(P^0(X), \supseteq, V)$. Write $Z \subseteq^0 Y$ ($Z \subset^0 Y$) to mean that Z is a non-empty (proper) subset of Y. To each set, $X = \{1, ..., n\}$, associate in some canonical way a propositional partition $A_1 ... A_n$. For non-empty $Y \subseteq X$, we will define a formula D^X_Y (or simply D_Y when X is clear from context). When $|Y| = 1$:

$$D_{\{m\}} = \Diamond\Box A_m \wedge \bigwedge_{k \neq m} \neg\Diamond\Box A_k$$

When D_Z is defined for $|Z| \leq k$ and $|Y| = k + 1$:

$D_Y = \bigwedge_{Z\subset^0 Y} \Diamond D_Z \wedge \Box(\bigwedge_{Z\subset^0 Y} \Diamond D_Z \vee \bigvee_{Z\subset^0 Y} D_Z)$.

The subset principle is then:[22]

Sub $\bigvee_{Y\in P^0(X)} D_Y$

Observe that Sub implies the formula $\Box(\bigwedge_{Z\subset^0 Y} \Diamond D_Z \vee \bigvee_{Z\subset^0 Y} D_Z)$ for each non-empty $Y \subseteq X$, and indeed, these formulas provide an equivalent formulation of Sub.

Proposition 30. *Every instance of* Sub *is in* Med.

Proof. Consider a Medvedev model $(P^0(Y), \supseteq, V)$, and for each $w \in P^0(Y)$ let $f(w) = \{k \in X \mid w \Vdash \Diamond\Box A_k\}$ for $X = \{1, 2, ..., n\}$.

Claim: $w \Vdash D_{f(w)}$

The proof is by induction on the cardinality of $f(w)$. If $|f(w)| = 1$ then $f(w) = \{m\}$ for some $m \in X$, so $w \Vdash \Diamond\Box A_m$ and $w \Vdash \neg\Diamond\Box A_k$ for $m \neq k$. That is, $w \Vdash D_{\{m\}}$.

Suppose the claim is true when $|f(w)| \leq k$. We must now show:

1. $w \Vdash \Diamond D_Z$ for each $Z \subset f(w)$
2. $w \Vdash \Box(\bigwedge_{Z\subset^0 f(w)} \Diamond D_Z \vee \bigvee_{Z\subset^0 f(w)} D_Z)$

For 1, note that for any $Z \subset^0 f(w)$, w sees at least a world v such that $f(v) = Z$, namely $\{a \in w \mid \exists k \in Z, \{a\} \Vdash A_k\}$. By the inductive hypothesis $v \Vdash D_{f(v)}$ where $f(v) = Z$, and so $w \Vdash \Diamond D_Z$.

For 2, suppose that $w \supseteq v$. If $f(v) = f(w)$ then $w \Vdash \bigwedge_{Z\subset^0 f(w)} \Diamond D_Z$ by the previous argument. Otherwise $f(v) \subset f(w)$, and by the inductive hypothesis $v \Vdash D_{f(v)}$, and so $v \Vdash \bigvee_{Z\subset^0 f(w)} D_Z$. $\Box$

Note that f is a p-morphism from the present Medvedev frame on $P^0(Y)$ to the frame on $P^0(X)$.

It may also be shown that all instances of Sub are valid for the full substitution class:

Proposition 31. Sub $\in L(S(\neg \wedge \Box))$.

Proof. Let $X = \{1, ..., n\}$, and let v be an arbitrary valuation of the full substitution class. We will show that $v(iD_{f(i)}) = 1$. It follows that $v(i \bigvee_{Y\subseteq X} D_Y) = 1$ for every substitution i, securing the validity of Sub.

Base: suppose $|f(i)| = 1$. So $f(i) = \{m\}$, which means that $v(\Diamond\Box iA_m) = 1$ and $v(\Diamond\Box iA_r) = 0$ for $r \neq m$.

Inductive Step: suppose the inductive hypothesis holds when $|f(i)| \leq m$, and suppose that $|f(i)| = m + 1$.

We must show

(i) $v(\Diamond iD_Z) = 1$ for each $Z \subset f(i)$ and

(ii) $v(\Box(\bigwedge_{Z\subset f(i)} \Diamond iD_Z \vee \bigvee_{Z\subset f(i)} iD_Z)) = 1$.

[22] It can be see to be a generalization of an axiom schema from Holliday (2017) and Hamkins et al. (2015), where $1 < k \leq m$:

$$(\bigwedge_{i\leq m} \Diamond\Box A_i \wedge \neg\Diamond \bigvee_{i\neq j} A_i \wedge A_j) \to \Diamond(\bigwedge_{i\leq k-1} \Diamond\Box A_i \wedge \bigwedge_{k\leq j\leq m} \neg\Diamond\Box A_j)$$

Our axiom is strictly stronger than this axiom in the presence of S4.

For (i), we may appeal to Lemma 10 to obtain a j such that $f(j \circ i) = Z$. Since $|f(j \circ i)| \leq n$ we may apply the inductive hypothesis and conclude $v(jiD_{f(j\circ i)}) = 1$, and thus that $v(\Diamond iD_{f(j\circ i)}) = 1 = v(\Diamond iD_Z)$.

Now we show (ii). For any j, $f(j \circ i) \subseteq f(i)$. If $f(j \circ i) = f(i)$ then $v(\Diamond jiD_Z) = 1$ for each $Z \subset f(i) = f(j \circ i)$ by repeating the reasoning for part (i) (using $j \circ i$ instead of i). If $f(j \circ i) \subset f(i)$ then $v(jiD_{f(j\circ i)}) = 1$ by the inductive hypothesis. So for any substitution j, $v(j(\bigwedge_{Z\subset f(i)} \Diamond iD_Z \vee \bigvee_{Z\subset f(i)} iD_Z)) = 1$ and so $v(\Box(\bigwedge_{Z\subset f(i)} \Diamond iD_Z \vee \bigvee_{Z\subset f(i)} iD_Z)) = 1$. $\Box$

Thus we have a tight bound on the logic of the full substitution class:

$\mathsf{S4MSub} \subseteq L(S(\neg \wedge \Box)) \subseteq \mathsf{Med}$.

There are, however, further valid principles of Med that we have not been able to verify to be valid for the full substitution class. One is:

(A1) $D_Y \to \Diamond(D_Y \wedge \bigwedge_{Z\subseteq^0 Y} \bigvee_k \Box(D_Z \to A_k))$

where $A_1...A_r$ is an arbitrary partition of propositional formulas.

(A1) may be seen to be valid as follows. Let $(P^0(Y), \supseteq, V)$ be a model over a Medvedev frame. Suppose D_Z is true at w, where $Z \subseteq X = \{1, ..., n\}$. The formulas $A_1...A_n$ partition the terminal worlds $\{\{m\} \mid m \in w\}$ according to which A_k they make true. For each formula A_k of $A_1...A_n$ such that $w \Vdash \Diamond\Box A_k$, pick a single representative $m_k \in w$, such that $\{m_k\} \Vdash A_k$. It is clear that w sees $\{m_1...m_n\}$, and that for any $Z \subseteq \{1, ..., n\}$, D_Z is true only at the corresponding subset of $\{m_1, ..., m_n\}$: $\{m_k \mid k \in Z\}$. Because each D_Z is true in exactly one world seen by $\{m_1, ..., m_n\}$, it follows that $\{m_1, ..., m_n\} \Vdash D_Y \wedge \Box \bigwedge_{Z\subseteq^0 Y} \bigvee_k \Box(D_Z \to B_k)$.

Another is a rule under which Med is closed:

(R1) If $\vdash (D^X_X \wedge \bigwedge_{Z\subseteq^0 X} \bigvee_k \Box(D^X_Z \to S_k)) \to C$ for for every finite set X, then $\vdash C$

Where S_k range over all possible consistent conjunctions of literals in the letters appearing in C and D_X. Indeed, it may be shown, without too much difficulty, that Med is the smallest modal logic containing S4, Sub and closed under (R1). Thus if we could show that Δ_v is closed under (R1) whenever v is a valuation of the full substitution class, the uniqueness conjecture would be solved.

We may also use the fact that Sub belongs to the logic of the full substitution class to fully resolve Humberstone's second question: the logic of the full substitution class is different from the logic of H.

Proposition 32. *$L(H)$ does not contain all instances of Sub.*

Proof. Let $A_1...A_4$ be the four way partition of propositional formulas: $p_1 \wedge p_2$, $p_1 \wedge \neg p_2$, $\neg p_1 \wedge p_2$, and $\neg p_1 \wedge \neg p_2$.

In any H-valuation, it is easy to see that $p_1 \wedge p_2$, $p_1 \wedge \neg p_2$, $\neg p_1 \wedge p_2$, and $\neg p_1 \wedge \neg p_2$ are all possibly necessary by considering the four possible ways of substituting $\top$ and $\bot$ for p_1 and p_2. Thus $v(D_Y) = 0$ for any proper subset Y of $\{1, 2, 3, 4\}$ and valuation v. So if Sub were valid, then $v(D_{\{1,2,3,4\}}) = 1$; and so, in particular, $v(\Diamond D_{\{1,4\}}) = 1$ by the definition of $D_{\{1,2,3,4\}}$.

But $\Diamond D_{\{1,4\}}$ implies $\Diamond\Box(p_1 \wedge p_2)$ and $\Diamond\Box(\neg p_1 \wedge \neg p_2)$, and also $\neg\Diamond\Box(p_1 \wedge \neg p_2)$ and $\neg\Diamond\Box(\neg p_1 \wedge p_2)$. Yet any H substitution according to which $p_1 \wedge p_2$ is possibly necessary and $\neg p_1 \wedge \neg p_2$ is possibly necessary, p_1 and p_2 must be mapped to themselves, and not to $\top$ or $\bot$, since it must remain possible to change the truth values of both p_1 and p_2. But relative to any substitution in which p_1 and p_2 are mapped to themselves, $\neg p_1 \wedge p_2$ and $p_1 \wedge \neg p_2$ must also be possibly necessary, since p_1 and p_2 may be replaced with $\top$ or $\bot$ at will. $\Box$

We end this section with a weaker conjecture. It's interesting to note that in the $S(\neg\wedge\Box)$-valuations that we constructed by setting $v(\Box A) = 1$ iff $A \in \mathsf{Med}$, every possibility is witnessed by a substitution of sentence letters with sentences of modal degree 2 (the sentences D^i_p) — for short, a substitution of modal degree 2. So if v is such a valuation, and if $v(\Diamond A) = 1$ then there is some substitution i of modal degree 2 such that $v(iA) = 1$. (Note that the substitutions of modal degree 2 do not themselves form a substitution class, since they are not closed under composition, despite the fact that they are well-behaved in this context.) This suggests the following more general claim:

Conjecture 33. *If v is a $S(\neg\wedge\Box)$-valuation and $v(\Diamond A) = 1$, then $v(iA) = 1$ for some substitution of modal degree 2.*

Given the above observation, this claim clearly follows from Conjecture 4, but is a question that might be more easily tackled directly.

5 Other Theories of Logical Necessity

In the introduction we considered a couple of alternatives to McKinsey's constraint. The first replaces the substitutional analysis of logical truth with an arbitrary logic Δ. The propositional analogue of logical truth, so conceived, is an interpretation of the modal operator $\Box$ under which a sentence $\Box A$ is true just in case A is a member of Δ. In the second, we replaced the Bolzanoean substitutional analysis with a Tarski-style analysis in a higher-order logic, in which a sentence $A(c_1...c_n)$ is logically true just in case the $\forall x_1...x_n.A(x_1...x_n)$ is true where $c_1...c_n$ enumerate the non-logical constants appearing in A.[23] The corresponding constraint on logical necessity can then by stated by a single biconditional:

$$\Box A(c_1...c_n) \leftrightarrow \forall x_1...x_n.A(x_1...x_n)$$

In this section we will investigate both of these options.

5.1 The Tarskian Constraint

According to the Tarskian analysis, one can express the logical truth of a sentence of a propositional modal logic, such as $\Box p \to p$, with a single sentence of propositionally quantified modal logic, in this case $\forall X(\Box X \to X)$. More generally, the logical truth of a closed sentence $A(p_1...p_n)$ in sentential constants $p_1...p_n$ should be equivalent to the truth of the quantified sentence $\forall X_1...X_n.A(X_1...X_n)$; and so one might articulate the idea that a propositional operator expresses the worldly analogue of logical truth under this conception by means of the following schema:

$$\Box A(p_1...p_n) \leftrightarrow \forall X_1...X_n.A(X_1...X_n)$$

This is clearly a restriction of what we earlier called the Tarskian Constraint to a sublanguage of higher-order logic.

[23] Tarski's theory of logical truth was originally formulated in a higher-order logic, as opposed to the set-theoretic model theory it has come to be associated with. The definition used here can be found earlier in Bernays and Schönfinkel (1928, p.347), and a related definition is found in Hilbert and Ackermann (1928, ch.IV §1). See also the notion of 'Metaphysical Universality' from Williamson (2013).

We will find it helpful to consider the dualized form of this schema:

$$\Diamond A(p_1 ... p_n) \leftrightarrow \exists X_1 ... X_n . A(X_1 ... X_n)$$

employing standard definitions for $\exists$ and $\Diamond$.

This schema may be regarded as expressing a principle of recombinatorialism with respect to the atomic propositions, a position once popular among the early logical atomists. For let us suppose that the sentential constants $p_1, ..., p_n$ of the language express atomic propositions. Since $A(X_1 ... X_n)$ contains propositional variables, not sentential constants, it expresses a relation among propositions defined in purely logical terms. So the right-to-left direction of the equivalence says that the particular atomic propositions $p_1, ..., p_n$ can instantiate any logical pattern that is in fact instantiated by some propositions, and the left-to-right direction tells us that these exhaust the logical patterns that the atomic propositions can instantiate. For instance, since there are truths and falsehoods, two instances of the schema ($\Diamond p_0 \leftrightarrow \exists X.X$ and $\Diamond \neg p_0 \leftrightarrow \exists X.\neg X$) tell us that every atomic proposition must be contingent. Similarly, since there are logically necessary propositions and logically false propositions, we may infer that every atomic proposition is possibly necessary and possibly necessarily false: $\Diamond \Box p_0$ and $\Diamond \Box \neg p_0$.[24] It is also worth noting that for this very reason we should not expect the schema itself to be necessary: we have already established that it is possible that p_0 is necessary, and also that certain instances of the schema imply that p_0 is contingent and so if all instances of the schema were necessary it would be possible for p_0 to be both necessary and contingent.

Here we show that there are indeed interpretations of propositionally quantified modal logic under which the schema is true. In the rest of this section we let $\mathcal{L}$ refer to the language of propositionally quantified modal logic, with the logical constants $\neg, \wedge, \forall, \Box$ and with infinitely many sentence letters. This language augments the language of propositional modal logic with an infinite set of variables that may occupy sentence position, $X_1, X_2, ...$, and a quantifier $\forall$ that can bind sentence variables. In addition to the usual syntactic clauses for propositional modal logic we also stipulate that sentence variables are formulas, and that $\forall X_k.A$ is a formula whenever A is.

We will consider the "full" Kripke models for this language in which the propositional quantifiers range over all sets of worlds (see Fine (1970)). Given an ordinary Kripke model for propositional modal logic, (W, R, V), we interpret an arbitrary formula of propositionally quantified modal logic as follows. Let g and g' range over variable assignments mapping each propositional variable X_k to a subset of W:

- $V^g(w, p_k) = V(w, p_k)$
- $V^g(w, X_k) = 1$ iff $w \in g(X_k)$
- $V^g(w, A \wedge B) = min(V^g(w, A), V^g(w, B))$
- $V^g(w, \neg A) = 1 - V^g(w, A)$
- $V^g(w, \Box A) = 1$ iff $V^g(w', A) = 1$ for every w' such that Rww'.

[24] The existence of truths and falsehoods may be derived given the usual axioms for the propositional quantifiers. E.g. existential generalization lets one move from $\top$ to $\exists X.X$ and $\neg\bot$ to $\exists X.\neg X$. The existence of logically necessary propositions and logically false propositions can also be derived directly: $\Box\forall X(X \vee \neg X) \leftrightarrow \forall X(X \vee \neg X)$ is an instance of the schema (where $n = 0$), and since the usual quantificational axioms secure $\forall X(X \vee \neg X)$, we may infer $\Box\forall X(X \vee \neg X)$ from which we may conclude $\exists X \Box X$ by existential generalization. The existence of logically false propositions follows by a similar argument.

- $V^g(w, \forall X_k A) = 1$ iff $V^{g'}(w, A) = 1$ for every g' for which g and g' agree on every variable except possibly on X_k

We may now construct a model of the Tarskian constraint.[25]

Theorem 34. *There is a model of the Tarskian constraint.*

Proof. Consider the Kripke frame:

- $W := \mathbb{N}^{<\omega}$ (finite sequences of naturals)
- Rww' iff w is a (proper or improper) initial segment of w'.

The frame can be visualised as an infinitary tree, that has the empty sequence as its root, and that branches infinitely many times at each node. Enumerate the sentences satisfiable at the root of this frame – A_1, A_2, A_3.... For each n, let (W, R, V_n) be a Kripke model over this frame in which A_n is true at the root. We paste these models together to make a single model over (W, R) as follows:

- $V((), p_i)$ may be set arbitrarily
- $V((n_1, ..., n_k), p_i) = V_{n_1}((n_2, ..., n_k), p_i)$ when $k \geq 1$

It is clear by construction that A_n is true at the world (n), and thus $\Diamond A_n$ is true at the root world $()$ for every n. Let $\mathcal{F} \uparrow w$ be the subframe of $\mathcal{F}$ generated by w. Then it is also clear that $(W, R) \uparrow w$ is isomorphic to (W, R) for any world $w \in W$.

Now if $\forall X_1...X_n.A(X_1...X_n)$ is true at the root world $()$ then A is valid over the frame (W, R). But since for any world $w \in W$, the generated frame $(W, R) \uparrow w$ is isomorphic to (W, R), it follows that A is valid over the frame $(W, R) \uparrow w$, and so true at the submodel of (W, R, V) generated by w. This means that A is true at w in (W, R, V), and since w was arbitrary, $\Box A$ is true at the base world $()$. Conversely, if $\Box A$ is true at $()$, then A must be true at (n) for each $n \in \mathbb{N}$, and thus $\neg A$ is not satisfiable in (W, R). So $\forall X_1...X_n.A[X_1/p_1...X_n/p_n])$ is true at $()$. $\Box$

We end with a conjecture that we have not been able to verify:

Conjecture 35. *There exists a model of the Tarskian constraint in $\mathcal{L}$ in which the propositional quantifiers are interpreted substitutionally.*

We make the conjecture precise as follows: there exists a valuation $v : \mathcal{L} \to \{0, 1\}$ such that (i) $v(\neg A) = 1 - v(A)$, (ii) $v(A \wedge B) = min(v(A), v(B))$, (iii) $v(\forall p.A) = 1$ iff $v(A[B/p]) = 1$ for every closed sentence B of $\mathcal{L}$ and (iv) $v(\Box A(p_1...p_n) \leftrightarrow \forall X_1...X_n.A(X_1...X_n)) = 1$ for every sentence A in the letters $p_1, ..., p_n$.

Given the substitutional interpretation of the quantifiers, the McKinseyan and Tarskian requirements coincide: it is easily verified that the conjecture is true if and only if there exists a valuation $v : \mathcal{L} \to \{0, 1\}$ satisfying (i)-(iii) that additionally satisfies McKinsey's requirement for the full substitution class of this language: $v(\Box A) = 1$ iff $v(iA) = 1$ for every substitution i mapping sentence letters to arbitrary closed sentences of $\mathcal{L}$. The substitutional intepretation of the quantifiers corresponds in a natural way to the idea found in early Wittgenstein and Russell that every proposition may ultimately be analyzed in terms of the logically atomic propositions and logical operations.

[25] This sort of construction can be generalized to all of higher-order logic — for details see the appendix of Bacon (2020). As opposed to here, the construction there is presented in terms of 'metaphysical substitutions' (see Bacon (2019) and also Fine (1977)) as opposed to Kripke models.

5.2 The Metalogical Constraint

We observed previously that in the valuation constructed in Theorem 9, $v(\Box A) = 1$ if and only if A was a member of a certain modal logic, Med. Let $\Delta_v = \{A \mid v(\Box A) = 1\}$. Then we have the following general result:

Proposition 36. *If v is a $S(\neg \wedge \Box)$-valuation then Δ_v is a modal logic extending $\mathsf{S4M}$.*

Proof. Propositions 14 and 15 already establish that whenever A is an instance of K, T, $\mathsf{4}$ or M then A is valid in v, i.e. $v(iA) = 1$ for every substitution $i \in S(\neg \wedge \Box)$ and, since v is a $S(\neg \wedge \Box)$-valuation, it follows that $v(\Box A) = 1$, and so $A \in \Delta_v$. Since every instance of $\mathsf{4}$ is true in v, Δ_v is closed under the rule of necessitation. Finally, Δ_v is closed under the rule of substitution since if $v(\Box A) = 1$, then $v(\Box\Box A) = 1$, and so $v(\Box iA) = 1$ for any substitution $i \in S(\neg \wedge \Box)$. $\Box$

Note that for an arbitrary substitution class, S, and S-valuation v, Δ_v will contain all instances of K, T, $\mathsf{4}$ and will be closed under the rule of necessitation. However, it may not be a modal logic because it may not be closed under the rule of substitution, but will only be guaranteed to be closed under substitutions in S.

It follows that the interpretation of $\Box$ in a $S(\neg \wedge \Box)$-valuation will satisfy an instance of the Meta-Logical Constraint considered in the introduction, according to which the interpretation of logical necessity is identified with a fixed modal logic Δ. We make this precise as follows:

Definition 9 (Meta-valuations)**.** *Let Δ be a normal modal logic. A function $v : \mathcal{L}(\neg \wedge \Box) \to \{0, 1\}$ is a Δ-valuation if and only if*

- $v(A \wedge B) = \min(v(A), v(B))$
- $v(\neg A) = 1 - v(A)$
- $v(\Box A) = 1$ *iff* $A \in \Delta$

As with the substitutional constraint, it is easily checked that whenever Δ is a consistent normal modal logic, then $\mathsf{4}$ must be true in a Δ-valuation, and moreover, the left-to-right direction of McKinsey's constraint is satisfied.

Any modal logic may, of course, be plugged into this definition. But if Δ is supposed to represent the logical truths and v a possible interpretation of $\Box$ as logical truth, then we would like the truths under the interpretation to include the logical truths: $v(A) = 1$ whenever $A \in \Delta$. Following Meyer (1971) we say:

Definition 10. *A modal logic Δ is coherent iff for every $A \in \Delta$ and Δ-valuation v, $v(A) = 1$.*

Intuitively, a modal logic is coherent when it accommodates an interpretation of $\Box$ with its own logic. Not every modal logic is coherent. Indeed, we can see that coherent modal logics cannot contain the B axiom (much as McKinsey's constraint rules out the Brouwerian axiom, except in degenerate cases). For suppose, for reductio, that Δ is a consistent modal logic and that v is a Δ-valuation in which every instance of B holds. For a sentence letter, p, $v(p) = 1$ or $v(\neg p) = 1$, so that either $v(\Box\Diamond p) = 1$ or $v(\Box\Diamond\neg p) = 1$. If the former, then $\Diamond p \in \Delta$, and since Δ is closed under the rule of substitution, $\Diamond\bot \in \Delta$, contradicting the assumption that Δ is a consistent modal logic. In the latter, we may similarly conclude $\Diamond\neg\top \in \Delta$.

Coherence is closely related to the disjunction property. In fact, it can be shown to be equivalent to an extended version of the disjunction property, which is satisfied by a normal modal logic Δ when the following holds for any sentences $A_1 ... A_n$ and any non-modal sentence A_0:

If $A_0 \vee \Box A_1 \vee ... \vee \Box A_n \in \Delta$ then $A_k \in \Delta$ for some k with $0 \leq k \leq n$.

We do not know of any normal modal logics possessing the disjunction property but not this extended version; we leave it as an open question whether it is indeed stronger.

Proposition 37. *A normal modal logic is coherent iff it has the extended disjunction property.*

Proof. Let Δ be a normal modal logic. Suppose that Δ is coherent, and $A_0 \vee \Box A_1 \vee ... \vee \Box A_n \in \Delta$. Suppose that for each k with $1 \leq k \leq n$, $A_k \notin \Delta$. It suffices to show that $A_0 \in \Delta$. From our first supposition it follows that $v(A_0 \vee \Box A_1 \vee ... \vee \Box A_n) = 1$, for any given Δ-valuation v. From our second supposition it follows that $v(\Box A_1) = ... = v(\Box A_n) = 0$; and so $v(A_0) = 1$. Since v was arbitrary, A_0 must be true in every Δ-valuation, and since A_0 is a propositional formula it must be a tautology.

Now suppose that Δ has the extended disjunction property. We will show that if a formula A is true in some Δ-valuation, v, then A is consistent with Δ, from which it follows that Δ is coherent. Suppose A is a truth functional combination of a formulas of the form $\Box B$ and propositional letters $p_1 ... p_n$. Let $\Box B_1 ... \Box B_m$ be the true formulas of the form $\Box B$, and $\Box C_1 ... \Box C_k$ the false formulas of this form. Finally, let D be the conjunction $\bigwedge_{v(p_i)=1} p_i \wedge \bigwedge_{v(p_i)=0} \neg p_i$. Clearly $v(D) = 1$ so D is truth-functionally consistent. So $\neg D \notin \Delta$. By the definition of a Δ-valuation $B_1 \in \Delta, ..., B_m \in \Delta$, and $\neg C_1 \notin \Delta, ..., \neg C_k \notin \Delta$. Since Δ has the extended disjunction property $(\neg D \vee \Box \neg C_1 \vee ... \vee \Box \neg C_n) \notin \Delta$. So the set $\{D, \neg \Box \neg C_1, ..., \neg \Box \neg C_n, \Box B_1, ..., \Box B_m\}$ is consistent with Δ, and since this set entails A, A too is consistent with Δ.

$\Box$

Remark 1. *Fine MS has considered a wider class of metavaluations obtained by relaxing the requirement that Δ be closed under the rule of substitution and/or necessitation. With the rule of substitution relaxed, we may find non-trivial valuations that validate* B. *(For instance by letting Δ be the set of C-validities; see section 3.2.)*

If Δ is not closed under rule of necessitation, the S4 *axiom is no longer guaranteed to hold. Urquhart (2010) investigates the result of letting Δ be the set of substitutions instances of tautologies, yielding an interpretation of logical necessity corresponding to tautologousness, which does not satisfy* S4.

Which normal modal logics are coherent? We may see immediately that if v is a valuation of the full substitution class, then Δ_v is a coherent modal logic.

Proposition 38. *If v is a $S(\neg \wedge \Box)$-valuation, Δ_v is a coherent modal logic.*

Proof. Let $A \in \Delta_v$ and let u be any Δ_v valuation. Since $A \in \Delta_v$, $v(\Box A) = 1$, and so $v(iA) = 1$ for every substitution i. We may construct a substitution j such that $v(jA) = u(A)$ by letting jp_k be p_k if $v(p_k) = u(p_k)$ and be $\neg p_k$ otherwise (this is proved by induction). So $u(A) = v(jA) = 1$, as required. $\Box$

But many other modal logics are coherent. Here is a fairly general sufficient condition for coherence. Recall that a pointed frame is a Kripke frame (W, R) equipped with a designated world $w_0 \in W$ which bears the ancestral of R to every element of W. The logic of a class of pointed frames is the set of sentences true at the designated world in any model over a pointed frame. If Δ is the logic of a class of Kripke frames $\mathcal{C}$, in the ordinary sense, it is also the logic of a class of pointed frames, $\{\mathcal{F} \uparrow w \mid \mathcal{F} \in \mathcal{C}, w \text{ a world in } \mathcal{F}\}$, writing $\mathcal{F} \uparrow w$ for the subframe of $\mathcal{F}$ generated by w, and $w \uparrow$ to be the set $\{v \mid Rwv\}$ when R is the accessibility relation of $\mathcal{F}$.

Definition 11 (Disjoint p-morphic copies)**.** *A class $\mathcal{C}$ of pointed frames is closed under disjoint p-morphic copies iff for any pointed frames $\mathcal{F}_1, ..., \mathcal{F}_n \in \mathcal{C}$, there is another pointed frame $\mathcal{F} \in \mathcal{C}$, worlds $w_1 ... w_n$ accessible to the designated world of $\mathcal{F}$, and p-morphisms $f_i : \mathcal{F} \uparrow w_i \to \mathcal{F}_i$ for $i = 1...n$ such that $w_i \uparrow \cap w_j \uparrow = \emptyset$ when $i \neq j$, and $f_i(w_i)$ is the designated world of $\mathcal{F}_i$.*

Example 6 (Coalesced sums)**.** *Given disjoint pointed Kripke frames $\mathcal{F}_1 ... \mathcal{F}_n$, their coalesced sum is the pointed Kripke frame $\mathcal{F} = (W, R, w_0)$ where:*

- $W := W_1 \cup ... \cup W_n \cup \{w_0\}$
- $R := R_1 \cup ... \cup R_n \cup \{w_0\} \times W$

The designated worlds $w_i \in W_i$, and the identity mappings from $\mathcal{F} \uparrow w_i$ to $\mathcal{F}_i$ comprise the relevant p-morphisms.

Note that many properties of the component frames are inherited by the coalesced sum. For instance, if $\mathcal{F}_1 ... \mathcal{F}_n$ are some combination of the properties of being reflexive, serial or transitive, then $\mathcal{F}$ is also that combination of reflexive, serial or transitive. So these classes are all closed under taking disjoint p-morphic copies.

Example 7. *The singleton class consisting of the frame $(\mathbb{N}^{<\omega}, \leq)$ from Theorem 34 — finite sequences of naturals with the initial subsequence ordering — is closed under disjoint p-morphic copies, since it is a countably infinite coalesced sum of isomorphic copies of itself.*

Example 8 (The class of Medvedev frames)**.** *The class of Medvedev frames is closed under disjoint p-morphic copies.*

Suppose $(P^0(X_i), \supseteq)$ for $i = 1...n$ are Medvedev frames. We can suppose, without loss of generality that $X_i \cap Y_j = \emptyset$ when $i \neq j$. Then $(P^0(X_1 \cup ... \cup X_n), \supseteq)$ is a Medvedev frame, and the worlds $X_1, ..., X_n \in P(X_1 \cup ... \cup X_n)$ with the evident identity mappings provide p-morphisms into each component $(P^0(X_i), \supseteq)$.

Example 9 (The class of finite partial function frames)**.** *The class of finite partial function frames is closed under disjoint p-morphic copies.*

Suppose $(X_i \rightharpoonup \{\top, \bot\}, \subseteq)$ are finite partial function frames for $i = 1...n$. We can suppose, without loss of generality that $X_i \cap Y_j = \emptyset$ when $i \neq j$, and that no X_i contains any numbers. Then $((X_1 \cup ... \cup X_n \cup \{1, .., n\}) \rightharpoonup \{\top, \bot\}, \subseteq)$ is a finite partial function frame. For each $i = 1...n$, let w_i be any function that is defined everywhere except X_i, maps i to $\top$ and maps j to $\bot$ for $j \in 1, ..., n$ and $j \neq i$. Due to the last condition, no partial function can be above both w_i and w_j when $i \neq j$. Moreover, the mapping $w \mapsto w \restriction X_i$ is a p-morphism from the frame generated by w_i to $((X_1 \cup ... \cup X_n \cup \{1, .., n\}) \rightharpoonup \{\top, \bot\}, \subseteq)$.

Proposition 39. *If Δ is characterized by a class of pointed frames that are closed under disjoint p-morphic copies, then Δ is coherent.*

Proof. Suppose Δ the logic of a class of pointed frames $\mathcal{C}$ that is closed under disjoint p-morphic copies, and suppose that there is some Δ valuation u such that $u(A) = 1$. We will show that A is true in some model over a frame in $\mathcal{C}$. Suppose that $\Box B_1, ..., \Box B_n$ are the subformulas of A of the form $\Box B$ such that $v(\Box B) = 0$. So $\neg B_1 ... \neg B_n$ are Δ-consistent and thus true in models $M_1 ... M_n$ over frames $\mathcal{F}_1 ... \mathcal{F}_n \in \mathcal{C}$. We know there is a frame $\mathcal{F}$ containing worlds $w_1 ... w_n$ and p-morphisms $f_i : \mathcal{F} \uparrow w_i \to \mathcal{F}_i$. Let V be any valuation on $\mathcal{F}$ such that whenever w is above w_i, $V(w, p_k) = V_i(f_i(w), p_k)$, and $V(w, p_k) = u(p_k)$ when w is the designated world of $\mathcal{F}$. We know these assignments cannot conflict, since $w_i \uparrow \cap w_j \uparrow = \emptyset$ when $i \neq j$. A is a truth-functional combination of sentence letters and formulas of the form $\Box B$. We know that $V(w, p_k) = u(p_k)$ for each sentence letter. If $u(\Box B) = 1$ then $B \in \Delta$ so $V(w, \Box B) = 1$ since $\mathcal{F} \in \mathcal{C}$. If $u(\Box B) = 0$, and $\Box B$ is subformula of A, then by construction, w sees a world in which B is false, so $V(w, \Box B) = 0$. Thus $V(w, A) = u(A)$, as required. $\Box$

Appealing to examples 6, 8 and 9 we may immediately infer:

Corollary 40. *K, KT, S4, KD, K4D are coherent.*

Corollary 41. *Medvedev logic and the logic of finite partial functions are coherent.*

It also follows that the logic of the infinite tree $(\mathbb{N}^{<\omega}, \leq)$ is coherent, but since this is just S4 this observation is already implied by Corollary 40.

Say that a coherent logic, Δ, is *maximal* iff whenever $\Delta \subseteq \Delta'$ and Δ' is coherent, $\Delta = \Delta'$. A straightforward application of Zorn's lemma establishes that there exist maximal coherent logics. Here we show that Med is a maximal coherent logic using the formulas described in section 2.1.[26]

Theorem 42. Med *is a maximal coherent logic.*

Proof. Suppose that Δ is a coherent logic and Med $\subseteq \Delta$. To show the logics are identical, it suffices to show that anything consistent in Med is consistent in Δ.

Suppose that $C(p_1 ... p_m)$, in letters $p_1 ... p_m$, is consistent in Med and true at the root X of a Medvedev model $(P^0(X), \subseteq, V)$. Without loss of generality, we may suppose that $X = \{1, ..., n\}$. Fix a propositional partition $A_1 ... A_n$ and define the formulas D_Y, for $Y \subseteq^0 X$ as in section 4.2.

Since A_k is a consistent propositional formula, it has a tautologous substitution instance, A'_k. Thus $\Diamond\Box A'_k \in$ KD $\subseteq \Delta$. Since $\Diamond\Box A_k$ has a substitution instance that is consistent in Δ it is itself consistent in Δ. Since Δ has the disjunction property, it follows that $\Diamond\Box A_1 \wedge ... \wedge \Diamond\Box A_n$ must be consistent in Δ, for otherwise its negation, and thus the disjunction $\Box\Diamond\neg A_1 \vee ... \vee \Box\Diamond\neg A_n$ would be in Δ while none of its disjuncts are.

The formula $\Diamond\Box A_1 \wedge ... \wedge \Diamond\Box A_n \to D_X$ is a consequence of Sub and so belongs to Med and thus also Δ. Since the antecedent is consistent in Δ, D_X must be consistent as well. So D_X is true at the root of a transitive reflexive Kripke model (W, R, U) which verifies all the theorems of Δ.

Since Sub $\subseteq$ Med $\subseteq \Delta$ it follows that for every $w \in W$, there is some $Y \subseteq^0 X$ such that $U(w, D_Y) = 1$, and this Y is moreover unique because the D_Y are pairwise incompatible.

[26] This is closely related to the problem of finding maximal superintuitionistic logics with the disjunction property. Indeed, the intuitionistic analogue of Med has this property, (see Maksimova (1986), Chagrov (1992) and the references found therein). For a survey, including a discussion of the related properties for modal systems, see Chagrov and Zakharyashchev (1991).

In fact, it is easily seen that this Y is $\{k \mid U(w, \Diamond\Box A_k) = 1\}$. Thus define a function $f : W \to P^0(X)$ by setting $f(w) = \{k \mid U(w, \Diamond\Box A_k) = 1\}$. This function is a p-morphism from the frame (W, R) to $(P^0(X), \subseteq)$. If Rxy then everything possible at y is possible at x since R is transitive, so $f(x) \subseteq f(y)$. If $f(x) \subseteq Y$, then, since $U(x, D_{f(x)}) = 1$, $D_{f(x)}$ has $\Diamond D_Y$ as a conjunct, and so x sees a world y such that $U(y, D_Y) = 1$, and so $f(y) = Y$.

For each letter p_k we may define a formula D_{p_k} as $\bigvee_{V(Y,p_k)=1} D_Y$. It is clear by definition that $U(w, D_{p_k}) = V(f(w), p_k)$, and since f is a p-morphism, $U(w, C(D_{p_1}...D_{p_m})) = V(f(w), C(p_1...p_m))$. We know that D_X is true at the root w_0 of W, so $f(w_0) = X$ and thus $U(w_0, C(D_{p_1}...D_{p_m})) = V(X, C(p_1...p_m)) = 1$. So $C(p_1...p_m)$ has a Δ-consistent substitution instance, $C(D_{p_1}...D_{p_m})$, and so must itself be Δ-consistent.

$\Box$

6 The Substitutional Approach to Modal Predicate Logic

The present paper has focused on modal propositional logic. But it is natural also to consider the application of the substitutional approach to modal predicate logic. This raises some interesting questions, which provide a new slant on some familiar approaches to the interpretation of modal predicate logic. We shall briefly discuss these questions although they call for a far more detailed treatment.

The introduction of identity into the language brings to the forefront some difficulties for the logical interpretation of $\Box$ that are closely related to difficulties Quine raised for modal logic more generally.[27] Quine noted that certain notions, like analyticity and logical truth, are not closed under the substitution of co-referential terms. For instance, 'Hesperus' is co-referential with 'Phosphorus' yet, while 'Hesperus is Hesperus' is a logical truth, 'Hesperus is Phosphorus' is not. This trio of claims is perfectly consistent with Leibniz's law, for the names 'Hesperus' and 'Phosphorus' do not actually appear in the latter two assertions, but are mere lexicographic components of a quotation name for a sentence. But, for just this reason, Quine argued that we cannot move up a grade of modal involvement and introduce a sentence operator that stands to the world, so to speak, as analyticity or logical truth stands to language. Now Kripke (1972) famously argued that we should distinguish analyticity, logical truth, and the like from metaphysical necessity and that metaphysical necessity can be expressed by a sentential operator for which it is necessary that Hesperus is identical to Phosphorus. Indeed, the inconsistency of the corresponding trio of claims involving metaphysical necessity may be interpreted as a demonstration of the necessity of identicals from Leibniz's law (Kripke (1971, p.163)).

But these considerations take on a critical significance when our goal is to introduce a notion of *logical necessity* that stands to reality as logical truth stands to language, for we would then like the metalinguistic and metaphysical notions to line up. So how might it be maintained to be *logically* necessary that Hesperus is Phosphorus? If one could speak of the logical form of a proposition — construed as a composite of individuals, relations, and so on, as opposed to sentences or Fregean thoughts — the proposition that Hesperus is Phosphorus would involve the same object on either side of the identity relation and so any proposition with the same logical form would be true. The enquoted sentence, by contrast, does not involve the same name twice: the discrepancy arises because the structure of the language used to express the propositions does not perfectly reflect the structure of the underlying reality — we have two different names for the same object. We ensured

[27]See, for example, *Reference and Modality* in Quine (1953).

that this problem wouldn't arise when stating the substitutional, metalogical and Tarskian constraints on logical necessity, when we insisted that we only accept the instances of these schemas formulated in a logically perfect language in Russell's sense; a language in which non-logical constants do not denote logically complex properties and no two non-logical constants co-denote. In this section we will show, among other things, how one might extend our treatment of logical necessity to logically imperfect languages.

An *interpretation* for a first-order language (without modality) may be taken to consist of a non-empty domain D of individuals and a function δ which assigns an individual from D to each individual constant and a subset of D^n to each n-place non-logical predicate letter. We shall assume, as usual, that $=$ is assigned the identity relation on D. We shall, for convenience, suppose that the models are *full* in the sense that for each individual $d \in D$ there is an individual constant a for which $\delta(a) = d$ — i.e. δ is surjective on individuals. An interpreted first-order language is *logically perfect* (with respect to the individual constants) when δ is injective.

The notion of a (concrete) substitution function and of its action on an arbitrary formula is now more complicated than in the propositional case. We should allow each individual constant to be replaced by another individual constant and we might also allow it to be replaced by a complex term should these be taken to be part of the original language (either through the use of a description operator or function letters). Similarly, we should allow each non-logical predicate letter to be replaced by another non-logical predicate letter, and we might also allow it to be replaced by any complex predicate expression. In order to indicate how a complex predicate expression is to apply to its arguments, we might indicate it by something like a λ-term $\lambda x_1 x_2 ... x_n.A$ and then, in making the substitution, we should take care to avoid any unintended clash of variables (a similar problem will also arise for complex individual terms should they be allowed to contain the description operator or other variable-binding operators). This means that we no longer have a simple compositional rule according to which for any substitution i:

$$i(Pt_1 t_2 \dots t_n) = iP(it_1 it_2 ... it_n)$$

since the result of substituting $\lambda x_1 x_2 ... x_n A(x_1, x_2, ..., x_n)$, for example, for P will be $A(t_1, ..., t_n)$ rather than $\lambda x_1 x_2 ... x_n A(x_1, x_2, ..., x_n) t_1 t_2 ... t_n$. We could, on the other hand, already allow complex predicate expressions to belong to the language and thereby avoid this difficulty, although at the expense of complicating the syntax of the language.

We can now bring to bear our earlier remarks concerning a Russellian logically perfect language. For the instances of McKinsey's schema — that '$\Box A$' is true iff every substitution instance of A is true — will depend on the language in which it is formulated. But as we have argued, in a logically imperfect language there may be instances of the schema that are false. For consider a valuation v for which $v(a = b) = 1$ and $v(a = c) = 0$ (where a and b are co-referential while a and c are not, as can happen in a logically imperfect language). Now $v(\Box(a = b)) = 1$ only if $v(i(a = b)) = 1$ for any allowable substitution i. But consider the substitution i of a for a and c for b. Then $v(i(a = b)) = v(a = c) = 0$; and so, if this substitution is allowed, $v(a = b \to \Box(a = b)) = 0$. But this violates the Necessity of Identity. It also leads to a violation of Leibniz's law, which we have suggested should be valid for any notion of propositional necessity. For substituting an arbitrary term for a in $a = a$ yields a truth. Hence $v(\Box a = a) = 1$ and $v(a = b \to \Box a = a \to \Box a = b) = 0$.

Of course, no ordinary language will be logically perfect. How then might we adopt a substitutional interpretation of necessity for such a language and yet still save Leibniz's law? One possible line of solution, considered in Fine (2005, pp.51-52, 109-110), is to introduce a notion of logical form of a sentence that takes into account the referents of the individuals constants. Thus 'Hesperus is Phosphorus' will be taken to be of a different logical form than 'Hesperus is Mars', since the former pair of terms are coreferential while the latter pair are not; and on the resulting conception of logical truth, 'Hesperus is Phosphorus' will count as a logical truth. What this means is that in logically imperfect languages, we should impose the constraint that substitutions preserve co-reference: if $\delta(a) = \delta(b)$ then $\delta(ia) = \delta(ib)$.[28] In a logically perfect language this constraint is vacuously satisfied, since $\delta(a) = \delta(b)$ only when a and b are the same term; and a requirement of this sort — in which semantical elements are allowed to make a contribution to logical form — is quite different from anything that arises in the propositional case.

Even with the assumption that we are working within a logically perfect language or with the above constraint on substitutions, we may still generate counterexamples to the necessity of distinctness:

$$a \neq b \to \Box a \neq b$$

For surely we can have two names for two different objects, a and b, whether our language is logically perfect or not. So $v(a \neq b) = 1$. Consider now the substitution i of a for a and of a for b. Then $v(i(a \neq b)) = v(a \neq a) = 0$, and so $v(\Box a \neq b) = 0$, assuming that i is indeed an allowable substitution.

One option here is to embrace these failures of the necessity of distinctness.[29] For unlike contingently true identities, contingently true distinctness claims are consistent with the usual logical axioms governing quantification and identity. It is only in the presence of Brouwer's principle, or one of its various weakenings, that the necessity of distinctness becomes a theorem of quantified modal logic (Prior (1967, p.146)).[30] Yet we have already considered an argument of McKinsey in section 3.1 that Brouwer's principle is not part of the logic of logical necessity. Perhaps we should say the same about the necessity of distinctness? (For there is, after all, a striking resemblance between McKinsey's argument and the above argument against the necessity of distinctness.)

We could, alternatively, ensure the necessity of distinctness by imposing additional constraints on substitutions. For just as we previously required that co-referential constants be substituted for co-referential constants, we could now require that non-co-referential constants be substituted for non-co-referential constants.[31] For we might wish to insist, at least in the case of the individual constants, that a substitution should preserve the exact logical form of the original expression — distinctions of logical form within the original expression should not be made to disappear. Thus, in the present case, a cannot be substituted for

[28]This corresponds to the requirement that the interpretation of identity should be persistent in a Kripke model of quantified $\mathsf{S4}$. More precisely, we may define a Kripke model (W, R, V) just as in Proposition 2 in which W is the set of substitutions, $R = \{(i, j \circ i) \mid i, j \in W\}$ and $V(i, Ra_1...a_n) = v(i(Ra_1...a_n))$ when R is any relation or predicate symbol (including identity). If $V(i, a = b) = 1$, then $v(i(a = b)) = 1$ so $\delta(ia) = \delta(ib)$, and since all substitutions preserve co-reference, $\delta(jia) = \delta(jib)$ for any $j \in W$. So $v(ji(a = b)) = V(j \circ i, a = b) = 1$, which means that identity is persistent.

[29]See Bacon (2020, pp.554-556).

[30]The proof in Prior (1967) is attributed to E.J. Lemmon. An earlier proof of the necessity of distinctness within a the stronger background logic of $\mathsf{S5}$ may be found in Prior (1955, p.206), concerning which Prior writes 'the obscurity of the issue does leave $\mathsf{S5}$ under a measure of suspicion, and this must be set against our earlier argument from simplicity in its favour' (p.207). (The necessity of distinctness may also be derived from various weakenings of Brouwer's principle, such as $A \to \Box\Diamond\Diamond A$, or $\Box(A \to \Box A) \to \Diamond A \to A$. However, in the present substitutional setting in which the principles of $\mathsf{S4}$ are being granted, these variant principles are *not* strict weakenings of Brouwer's principle but equivalent to it.)

[31]This corresponds to the requirement that the interpretation of identity be anti-persistent in the Kripke model associated with a valuation v, i.e. if $j \circ i \Vdash a = b$ then $i \Vdash a = b$. See footnote 28 above.

both a and b since $a = b$ and $a = a$ are not of exactly the same logical form. And again, this is a requirement that has no counterpart - or, at least, no useful counterpart - in the propositional case.

We might wish to further restrict or modify the constraints on which substitutions on individual constants should be allowed. Let IC the set of individual constants and PL the set of predicate letters of the language in question. We may then represent a substitution as a partial function from $IC \cup PL$ subject to the requirement that individual constants go into individual terms and predicate letters go into predicate expressions. We may then suppose that we are given an arbitrary set S of allowable substitutions subject only to the requirement that the restriction of any substitution in S is also a substitution S. Since the functions in S are now partial, the clause for $\Box A$ should be modified accordingly, so that $v(\Box A) = 1$ iff $v(iA) = 1$ for all allowable substitutions i defined on A.

Further conditions of a general structural nature might then be imposed on S. It might be required, for example, that each partial function in S should be extendible to a total function or that the functions on individual constants and on predicate letters should be freely combinable, so that if i and j are two partial functions in S then the partial function $i \upharpoonright IC \cup j \upharpoonright PL$ should also be in S. But the former requirement might be questioned if it was thought that the individuals designated by the constants in the range of a substitution function should somehow be compossible; and even the latter requirement might be questioned if it was thought that a predicate letter P could contain a "hidden" individual constant a, so that a restriction on what could be substituted for a would result in a restriction on what could be substituted for P.

Constraints of a more particular character might also be imposed on S. It might be supposed, for example, that certain constants a (including the special constants that correspond to an individual a) should be "rigid", so that only a can be substituted for a. This would provide us with another way of securing the validity of the principle $a = b \to \Box(a = b)$ and of the principle $a \neq b \to \Box a \neq b$ in the case in which a and b are rigid constants, since the allowable substitutions would leave them intact. It would also allow us, given that certain non-logical predicates were allowed to be rigid, to sustain the truth of various substantive essentialist claims. If, for example, 'Felix' was a rigid constant for a tiger and 'tiger' a rigid predicate, then Felix would necessarily be a tiger ($v(\Box Tiger(Felix)) = 1$) while it might well be false that Felix is necessarily in a zoo (($v(\Box Zoo(Felix)) = 0$) given that the predicate 'Zoo' is not rigid, since a predicate P for which $P(Felix)$ is false might then be substituted for 'Zoo'. In this way, we could, to some extent, mimic an essential view of individual objects within the substitutional framework.

We could, in a similar way, mimic Lewis's counterpart theory. For when Lewis would say that a' is a counterpart of a, we might say that a′ can be substituted for a or when one might say, more generally, that a', b', c', ... are relational counterparts of a, b, c, ..., we might say that a′, b′, c′, ... can be simultaneously substituted for a, b, c, ... Our substitution-theoretic clause for necessity would then correspond to his counterpart-theoretic clause. We thereby obtain a highly non-realist version of counterpart theory, one that might be especially suited to someone who wished to adopt a linguistic view of necessity. We should also note that, just as we might wish the substituends for predicate letters with "hidden" individual constants to vary with those for the individual constants, so one might wish the counterparts for the non-qualitative properties expressed by predicate letters to vary with those for the individuals designated by individual constants, even though this was no part of Lewis' original proposal.[32]

[32]Dorr (2005) discusses a related difficulty which arises when the language allows us to quantify over

Quantifiers give rise to further issues. Within the present setting, it is natural to adopt a substitutional interpretation of the quantifier. Thus we can either say:

$v(\forall x A(x)) = 1$ iff $v(A(t)) = 1$ for all terms t

thereby obtaining something akin to a "conceptual" interpretation of the quantifier, or we can say:

$v(\forall x A(x)) = 1$ iff $v(A(t)) = 1$ for all rigid terms t,

thereby obtaining something akin to an "objectual" interpretation of the quantifiers. Let us restrict our attention to the objectual interpretation although some related issues also arise for the conceptual interpretation.

Then under the innocuous assumption that a rigid constant can always be substituted for itself, we can establish the equivalence of $\forall x \Box A(x)$ to $\Box \forall x A(x)$ ($v(\forall x \Box A(x)) = v(\Box \forall x A(x)$ for any valuation v). How might this equivalence be avoided, i.e. how might we gain the effect of a variable domain of quantification? One possibility is to suppose that our quantifiers are restricted. Thus universal quantifications now take the form $\forall x(A(x) : B(x))$ (all A's are B's) and we set $v(\forall x(A(x) : B(x))) = 1$ iff for every rigid term a either $v(A(a)) = 0$ or $v(A(a)) = 1$.

We now take the unrestricted quantification $\forall x A(x)$ to be implicitly restricted to a dummy domain predicate letter D. Thus $\forall x A(x)$ is read as $\forall x(Dx : B(x))$. This will then give us the effect of a variation in the domain, since different substitutions of a predicate P for D will provide us with different range of rigid constants a for which $v(Pa) = 1$. The equivalence of $\forall x \Box A(x)$ to $\Box \forall x A(x)$ will then fail in both directions. For $\forall x \Box A(x)$ will be read as $\forall x(Dx \to \Box A(x))$ while $\Box \forall x A(x)$ will be read as $\Box \forall x(Dx \to A(x))$; and as long as the domain predicate D is not rigid, neither formula will be equivalent to the other under a constant domain interpretation of the unrestricted quantifier.

As should be clear from this brief discussion, the substitutional approach has enormous flexibility in its application to modal predicate logic. It is able to accommodate a wide variety of interpretative stances - necessary versus contingent identity, objectual versus conceptual quantification, rigid versus counterpart-theoretic accounts of de re modality, fixed versus variable domains; and it may also be able to provide a certain degree of succor for those who favor a less metaphysical understanding of de re modality.

References

Alan R. Anderson and Nuel D. Belnap. *Entailment: The Logic of Relevance and Neccessity, Vol. I.* Princeton University Press, 1975.

Andrew Bacon. Substitution structures. *Journal of Philosophical Logic*, 48(6):1017–1075, 2019. doi: 10.1007/s10992-019-09505-z.

Andrew Bacon. Logical combinatorialism. *Philosophical Review*, 129(4):537–589, 2020. doi: 10.1215/00318108-8540944.

Paul Bernays and Moses Schönfinkel. Zum entscheidungsproblem der mathematischen logik. *Mathematische Annalen*, 99(1):342–372, 1928.

propositions but, if we are right, the difficulty can already be seen to arise within the original framework of modal first-order predicate logic.

Rudolf Carnap. Modalities and quantification. *Journal of Symbolic Logic*, 11(2):33–64, 1946. doi: 10.2307/2268610.

Rudolf Carnap. *Meaning and Necessity*. University of Chicago Press, 1947.

Michael J. Carroll. An axiomatization of s13. *Philosophia*, 8(2-3):381–382, 1978. doi: 10.1007/BF02379250.

Aleksandr Vasilevich Chagrov. Continuality of the set of maximal superintuitionistic logics with the disjunction property. *Mathematical Notes*, 51(2):188–193, 1992.

Alexander Chagrov and Michael Zakharyashchev. The disjunction property of intermediate propositional logics. *Studia Logica*, 50(2):189–216, 1991. doi: 10.1007/BF00370182.

Nino B. Cocchiarella. Logical atomism and modal logic. *Philosophia*, 4(1):41–66, 1974. doi: 10.1007/bf02381515.

M. J. Cresswell and G. E. Hughes. *A New Introduction to Modal Logic*. Routledge, 1996.

MJ Cresswell. Carnap and Mckinsey: Topics in the pre-history of possible-worlds semantics. In *Proceedings of the 12th Asian Logic Conference*, pages 53–75. World Scientific, 2013.

Cian Dorr. Propositions and counterpart theory. *Analysis*, 65(3):210–218, 2005. doi: 10.1111/j.1467-8284.2005.00551.x.

F. R. Drake. On Mckinsey's syntatical characterizations of systems of modal logic. *Journal of Symbolic Logic*, 27(4):400–406, 1962. doi: 10.2307/2964546.

Kit Fine. Modal logic as metalogic. unpublished.

Kit Fine. Propositional quantifiers in modal logic. *Theoria*, 36(3):336–346, 1970. doi: 10.1111/j.1755-2567.1970.tb00432.x.

Kit Fine. Properties, propositions and sets. *Journal of Philosophical Logic*, 6(1):135–191, 1977. doi: 10.1007/bf00262054.

Kit Fine. *Modality and Tense: Philosophical Papers*. Oxford University Press, 2005.

Harvey Friedman. One hundred and two problems in mathematical logic. *Journal of Symbolic Logic*, 40(2):113–129, 1975. doi: 10.2307/2271891.

Joel David Hamkins, George Leibman, and Benedikt Löwe. Structural connections between a forcing class and its modal logic. *Isr. J. Math.*, 201:617–651, 2015.

Herbert E. Hendry and M. L. Pokriefka. Carnapian extensions of S5. *Journal of Philosophical Logic*, 14(2):111–128, 1985. doi: 10.1007/BF00245990.

D Hilbert and W Ackermann. Grundzüge der theoretischen logik. *Julius Springer, Berlin*, 1928.

Wesley Holliday. On the modal logic of subset and superset: Tense logic over Medvedev frames. *Studia Logica*, 105(1):13–35, 2017. doi: 10.1007/s11225-016-9680-1.

Lloyd Humberstone. *Philosophical Applications of Modal Logic*. College Publications, 2016.

Saul Kripke. *Naming and Necessity*. Harvard University Press, 1972.

Saul A. Kripke. A completeness theorem in modal logic. *Journal of Symbolic Logic*, 24(1): 1–14, 1959. doi: 10.2307/2964568.

Saul A. Kripke. Identity and necessity. In Milton Karl Munitz, editor, *Identity and Individuation*, pages 135–164. New York: New York University Press, 1971.

Saul A. Kripke. Is there a problem about substitutional quantification? In Gareth Evans and John McDowell, editors, *Truth and Meaning*, pages 324–419. Oxford University Press, 1976.

Larisa L Maksimova. On maximal intermediate logics with the disjunction property. *Studia Logica*, 45(1):69–75, 1986.

J. C. C. McKinsey. On the syntactical construction of systems of modal logic. *Journal of Symbolic Logic*, 10(3):83–94, 1945. doi: 10.2307/2267027.

Robert K. Meyer. On coherence in modal logics. *Logique Et Analyse*, 14:658–668, 1971.

Prior. *Formal Logic*. Oxford University Press, 1955.

Arthur Prior. *Past, Present and Future*. Clarendon Press, 1967.

Tadeusz Prucnal. On two problems of harvey friedman. *Studia Logica*, 38(3):247–262, 1979. doi: 10.1007/BF00405383.

W. V. O. Quine. *From a Logical Point of View*. Harvard University Press, 1953.

Bertrand Russell. *The Philosophy of Logical Atomism*. Routledge, 1940.

Steven K. Thomason. A new representation of S5. *Notre Dame Journal of Formal Logic*, 14(2):281–284, 1973. doi: 10.1305/ndjfl/1093890907.

Alasdair Urquhart. Anderson and Belnap's Invitation to Sin. *Journal of Philosophical Logic*, 39:453–472, 2010.

Timothy Williamson. *Modal Logic as Metaphysics*. Oxford University Press, 2013.